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THE ASSIMILATION OF AMMONIUM NITROGEN BY NITZSCHIA CLOSTERIUM AND OTHER MARINE PHYTOPLANKTON

By Claude E. ZoBell

Scripps Institution of Oceanography, University of California, La Jolla, Calif.

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The origin of ammonia in the sea is a fairly well understood process but little is known concerning the formation of nitrate. It is the consensus of opinion of most oceanographers that nitrate nitrogen is essential for the production of marine vegetation, although experiments during the last two decades of many investigators (cf. reviews by Allison, 1931, and Pardo, 1935) have established that a large number of terrestrial plants assimilate ammonium nitrogen. Brandt (1929), Thompson and Robinson (1932), and others point out that oceanographic data including chemical analyses fail to indicate the utilization of ammonium nitrogen by phytoplankton. However, in discussing a more satisfactory method for the quantitative estimation of ammonia in sea water, Wattenberg (1929) suggests that it is advisable to devote more attention to this form of nitrogen as a possible nutrient for marine flora. Cooper's (1933) extensive investigations in the English channel indicate that ammonia may be directly utilized by phytoplankton. The following results furnish proof that certain bacteria-free cultures of phytoplankton do assimilate ammonium nitrogen.

Experimental.—The organisms were cultured in aged sea water to which was added 1.0 cc. per liter of a mineral solution essentially the same as that recommended by Allen (1913) plus a little silicate:

$Na_2HPO_4 \cdot 12H_2O$	5.0 gm.
$FeCl_3 \cdot 6H_2O$	2.0 gm.
$Na_2SiO_3 \cdot 9H_2O$	1.0 gm.
HCl (con.)	2.0 cc.
Distilled water	100.0 cc.

Like Harvey (1933), it was found that the silicate improved the growth of the diatoms and sometimes it became a limiting factor when sea water

was not enriched with it. The medium was heated to 70°C., cooled and the clear supernatant fluid aseptically decanted into sterile Pyrex Erlenmeyer flasks. Sterile solutions of various ammonium, nitrite and nitrate salts were added in different concentrations.

Most of the results reported below were obtained with a pure bacteria-free culture of *Nitzschia closterium*. Locally isolated cultures of *Nitzschia bilobata*, *Navicula* sp., *Chlorella* sp. and certain mixed cultures were also used. The test media were inoculated with uniform suspensions of the cultures to give approximately 500 cells per cc. after which they were exposed to indirect sunlight in north windows. The comparative rates of multiplication were followed by noting the first day on which growth became macroscopically visible and estimating the relative turbidity each day thereafter. Attempts were made to count the number of cells per unit of solution but their adherence to the glass and their tendency to form tenacious clumps under certain conditions interfered with direct microscopic enumeration. Frequent quantitative determinations of the

TABLE 1

COMPARATIVE ASSIMILATION OF AMMONIUM, NITRITE AND NITRATE BY *N. closterium* AS INDICATED BY GROWTH AND THE DISAPPEARANCE OF THE NITROGEN COMPOUNDS STARTING WITH 0.20 MILLIMOL OF EACH

TIME IN DAYS	AMMONIUM GROWTH	NH_4	NITRITE GROWTH	NO_2	NITRATE GROWTH	NO_3
0	−	0.20	−	0.20	−	0.20
4	*T*	0.19	?	0.20	−	0.18
7	+	0.19	*T*	0.20	*T*	0.16
10		0.17	*T*	0.18	+	0.16
14	++	0.16	++	0.17	++	0.15
18	+++	0.12	++	0.15	+++	0.14
21	+++	0.11	++	0.15	+++	0.10
28	+++	0.06	+++	0.09	++++	0.03
35	++++	0.01	+++	0.04	++++	0.001
42	++++	0	++++	0	++++	0

− no growth, *T* trace, + to ++++ increasing amounts of growth.

nitrogen compounds furnished another accurate index to the activity of the plankton in the different media. Nitrite was determined with the sulphanilic acid dimethyl-α-naphthylamine reagent, ammonia with Wattenberg's modified Nessler's solution and nitrate with diphenylbenzidine.

Table 1 illustrates the kind of record which was kept and shows that bacteria-free cultures of *N. closterium* assimilate ammonium and in so doing multiply more rapidly at first than in corresponding media containing equi-molar concentrations of either nitrite or nitrate. Similarly *Nitzschia bilobata*, *Navicula* sp., *Chlorella* sp. and certain mixed cultures of phytoplankton flourished on ammonium nitrogen, quantitatively exhausting as much as 0.2 millimol of this nitrogen compound

from the sea-water solutions. There were no macroscopic signs of multiplication in the controls to which no nitrogen was added although microscopic examinations revealed a two- to four-fold increase in the number of cells, probably attributable to a small quantity of residual available nitrogen in the sea water. Later experiments with synthetic sea-water controls demonstrated that the diatoms multiply with ammonium as the only source of nitrogen.

Provided small concentrations (0.2 millimol or less) of ammonium, nitrite and nitrate are compared, in virtually every instance multiplication and utilization first becomes manifest with ammonium. The initial superiority of ammonium vanishes in a week or two after which time there is little or no difference in the quantity of vegetation. The addition of a little ammonium to either nitrite or nitrate media accelerates multiplication. Combinations of ammonium and nitrate compounds are found to be better than either ion used separately. Ammonium nitrate in concentrations up to 0.1 millimol gave good results. The chloride, hydroxide, sulfate, phosphate or carbonate of ammonium supplied the nitrogen requirements of the diatoms.

TABLE 2

OPTIMUM AND INHIBITORY CONCENTRATIONS OF AMMONIUM, NITRITE AND NITRATE FOR THE MULTIPLICATION OF *N. closterium* AND THE CONCENTRATION OF THESE IONS REPORTED FOUND IN THE SEA

FORM OF NITROGEN	MILLIMOLS		MILLIMOLS IN SEA WATER	
	OPTIMUM	INHIBITORY	AVERAGE	RANGE
Ammonium	0.01–0.05	4	0.001–0.005	0–0.02
Nitrite	0.05–5.00	25	0.000,1	0–0.001
Nitrate	0.10–40.0	300	0.002–0.01	0–0.05

Undoubtedly some of the reported failures of phytoplankton and other plants to grow in ammonium media are due to the use of excessive quantities of ammonium. The effect of the concentration of ammonium, nitrate and nitrite nitrogen was studied by inoculating sea-water media containing the nitrogen compounds in different combinations with diatoms. The optimum concentration for multiplication was accepted as that in which in the majority of the cases vegetation first became perceptible to the naked eye and not necessarily that in which growth was best after prolonged incubation. The results for *N. closterium* are presented in table 2 which also expresses the quantity of these ions reported to be found in the sea. It will be observed that the optimum concentration of nitrate is over a hundred times greater than that for ammonium and also the optimum concentration of each nitrogen compound is greatly in excess of its average concentration found in the sea. In media which contained 0.002 millimol of either ammonium or nitrate nitrogen, approximately the average concentration which occurs in the sea, micro-

scopic examinations revealed appreciable multiplication with ammonium being superior at first, but later there was no detectable difference in the number of cells in the two substrata.

Tiedjens and Robbins (1931) confirm the findings of several other investigators that many crop plants preferentially assimilate ammonium at pH 7.0 or above whereas in acid media nitrate is more available, and at pH 4.0 ammonium is not assimilated. Efforts were made to ascertain the effect of the hydrogen-ion concentration upon the assimilation of ammonium by the marine phytoplankton. Dilute sodium hydroxide or hydrochloric acid was added to the sea-water media to give reactions

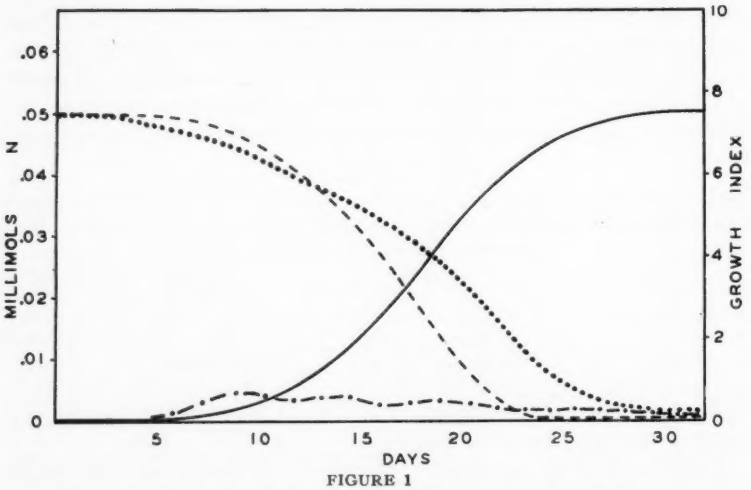

FIGURE 1

Depletion of nitrate (dotted line) and ammonium nitrogen (dashed line) accompanying the multiplication of *N. closterium* in sea-water medium which initially contained a mixture of 0.05 millimol of each. The appearance and disappearance of nitrate (dot-dash line) and the cumulation of vegetation (solid line) are also shown.

ranging by increments of 0.4 from pH 6.0 to 9.0. The most acid media in which *N. closterium* grew was pH 6.8 but the growth was so scant and abnormal that it was impossible to differentiate between the results with nitrate and ammonium. Since both forms of nitrogen supported multiplication in more alkaline media up to and including pH 8.8, these values are accepted as the limits. The optimum was found to be pH 7.6 to 8.0 although the sea-water habitat of these organisms is pH 8.0 to 8.4. There were no consistent differences in the assimilation of ammonium and nitrate at the different reactions.

Evidence for the preferential assimilation of ammonium in the presence

of nitrate is presented graphically in figure 1. These data were obtained by inoculating media containing 0.05 millimol each of ammonium and nitrate with pure cultures of *N. closterium*. At various intervals the amount of each ion and of vegetation was determined. Under these conditions the ammonium was depleted from the solution several days before the nitrate. Special attention is called to the appearance of a trace of nitrites.

Nitrites have been detected in small quantities in nitrate media seeded with *Nitzschia closterium*, *N. bilobata*, *Navicula* sp., *Chlorella* sp. and several unidentified mixed cultures of phytoplankton under conditions which are unfavorable to the activity of denitrifying bacteria. Depending primarily upon the size of the inoculum, a trace of nitrite begins to appear within three to seven days. The addition of a little ammonium to nitrate media usually increases the amount of nitrite which accumulates. Inasmuch as nitrite does not occur in ammonium medium lacking nitrate and does occur in nitrate media lacking ammonium, it is assumed that the diatoms reduce the nitrate to nitrite. During the initial positive acceleration growth phase the concentration of nitrite reaches its maximum and decreases with fluctuations during the logarithmic phase of growth. Finally the nitrite is exhausted at about the same time as the nitrate but due to the difficulties of determining nitrate in the presence of nitrite, it has not yet been possible to ascertain which disappears first or if they are simultaneously depleted from solution by the diatoms. It may be significant that in nitrate media teeming with multiplying diatoms, the concentration of nitrite is usually greatest early in the morning and diminishes with exposure to light. A few million diatoms per cc. were found to reduce a little nitrate to nitrite in total darkness and it is reiterated that rigorous cultural as well as direct microscopic tests proved the absence of bacteria.

Conclusions.—It is believed that the diatoms extracellularly reduce nitrate to nitrite in the process of its assimilation and eventual conversion into organic nitrogen. Beckwith (1933) demonstrated that certain pure cultures of *Chlorella* reduced nitrate to nitrite and that they assimilated the nitrite. Eckerson (1924) and others have presented experimental evidence to show that during protein synthesis by certain higher green plants nitrate is reduced to nitrite and then to ammonia. In view of the fact that protoplasmic nitrogen is in a reduced state, theoretically nitrite or ammonium nitrogen should be more readily available than nitrate nitrogen. It does not seem logical that all plant nitrogen must go through the roundabout process of being completely oxidized to nitrate only to be reduced again for conversion into amino nitrogen of protoplasm. It is more credible that particularly in the sea where the economy of Nature is at its best there is a short-cut from this part of the nitrogen cycle with nitrification serving primarily as a regulatory or storage process. Cooper

(1933) has shown that *Biddulphia mobiliensis* assimilates ammonia from sea water. According to Schreiber (1927) more *Carteria* cells were produced from ammonium than from an equivalent of nitrate. Harvey (1933) found that *Nitzschia closterium* grew just about as well in ammonium as in nitrate. Pure cultures of *Chlamydomonas* utilized ammonium more efficiently than nitrate in experiments described by Braarud and Föyn (1931) and Bond (1933) found that ammonium nitrogen satisfied the requirements of *Portotheca zopfii*, *Polytoma uvella*, *Chlamydomonas pulvinata* and *Dunaliella salina*.

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GROUPS IN WHICH THE SQUARES GENERATE A SUBGROUP OF INDEX LESS THAN SEVEN

BY G. A. MILLER

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS

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The squares of the operators of a given group G generate an invariant subgroup H and the quotient group G/H is an abelian group of order 2^α and of type $(1, 1, 1, \ldots)$. If a Sylow subgroup of order 2^m contained in G is cyclic but not the identity then G contains one and only one subgroup of index 2 and hence this subgroup is generated by the squares of its operators. The fact that such a G contains at least one subgroup of index 2 is a direct consequence of the obvious existence of negative substitutions therein when it is represented as a regular substitution group. It could not contain more than one such subgroup since the cross-cut of all of its subgroups of index 2 is invariant and gives rise to an abelian quotient group of order 2^α and of type $(1, 1, 1, \ldots)$. This invariant subgroup is evidently generated by the squares of the operators of G since these squares appear in every subgroup of index 2.

The fact that there are groups which separately involve only one subgroup of index 2 but do not contain a cyclic Sylow subgroup of order 2^m results directly from the symmetric groups of degree $n > 3$, since each of these groups is known to contain only one subgroup of index 2 but its Sylow subgroup of order 2^m is non-cyclic. The direct product of a simple group whose order exceeds 2 and the group of order 2 contains one and only one subgroup of index 2 and hence this subgroup is generated by the squares of its operators. Moreover, if such a simple group is extended by an operator so as to obtain a group of twice its order and if this extending operator transforms it according to an inner isomorphism then this extended group is such a direct product. That is, *if a group contains a simple group whose order exceeds* 2 *as a subgroup of index* 2 *generated by the squares of its operators but is not the direct product of this simple group and a group of order* 2 *then it is the extension of this simple group and an operator which transforms it according to an outer isomorphism whose order is a power of* 2.

A necessary and sufficient condition that the squares of the operators of a group of order 2^m generate a subgroup of index 2 is that this group is cyclic. In a non-cyclic group of order 2^m the subgroup generated by the squares of its operators therefore has an index which is at least 4. We proceed to determine some of the properties of those groups in which this index is exactly 4. A necessary and sufficient condition that this index is exactly 4 is that the group G contains exactly three subgroups of index 2.

Another necessary and sufficient condition is that G contains a set of two independent generators and hence its ϕ-subgroup is of index 4. We shall now prove that no one of the three subgroups of index 2 contained in G has more than three independent generators.

To prove this fact we shall first prove that G cannot contain an abelian subgroup of type $(1, 1, 1, \ldots)$ which contains as many as four independent generators. If such a subgroup appeared in G an additional operator of G would transform all the commutators arising from it into themselves and hence the order of the commutator subgroup could not exceed the square root of the order of this abelian subgroup. The squares of the additional operators of G would also appear in the central of G. Hence the commutators of G and the squares of the operators of G would generate a characteristic subgroup of G whose order could not exceed the double of the order of this commutator subgroup and hence it would be at least of index 8 under G. Consequently G would involve at least seven subgroups of index 2. As this is contrary to the hypothesis it has been proved that G cannot contain an abelian subgroup of index 2 and of type $(1, 1, 1, \ldots)$ which has as many as four independent generators.

If G involved another subgroup of index 2 containing at least as many as four independent generators then the squares of the operators of this subgroup would generate a characteristic subgroup of it which would give rise to an abelian quotient group of type $(1, 1, 1, \ldots)$ having at least as many as four independent generators. The reasons why this is impossible are obviously similar to those noted in the preceding paragraph and hence there results the following theorem: *If a group of order 2^m has the property that the squares of its operators generate a subgroup whose index does not exceed 4 then it cannot involve a subgroup of index 2 which has as many as four independent generators.* It therefore results that if in a group of order 2^m the squares of the operators generate a subgroup whose index does not exceed 4 then it involves no subgroup of index 2 which has more than three independent generators.

It is well known that when G involves a cyclic subgroup of index 2 there are five and only five groups of order 2^m, $m > 3$, in which the squares of the operators generate a subgroup which is exactly of index 4. Two of these groups involve separately two cyclic subgroups of index 2 while each of the remaining three involves only one such subgroup. In the latter cases each of the two additional subgroups of index 2 has two independent generators. Every quotient group of the groups under consideration has at most two independent generators. In particular, the possible non-cyclic quotient groups of order 8 are the quaternion group, the octic group and the abelian group of type (2,1).

To prove that the same G cannot have more than one of these groups of order 8 as a quotient group we may note that first if it had at least two

such quotient groups it would have at least three, which would not necessarily be distinct, since each such quotient group would involve a subgroup of order 2 corresponding to the subgroup of index 4 in G generated by the squares of its operators. Hence two invariant subgroups of index 8 which would give rise to these quotient groups would have an invariant cross-cut of index 16 under G which would give rise to the four group as a quotient group under the subgroup of the squares in G. The squares of some of the remaining operators of G would appear in every pair of cosets corresponding to operators of this four group. As this is obviously impossible it has been proved that *a group of order* 2^m *in which the squares of the operators generate a subgroup of index* 4 *is completely characterized by its non-cyclic quotient group of order* 8.

It is easy to prove that there are infinite systems of groups belonging to each of the three groups of order 8 having two independent generators. To construct such a system belonging to the quaternion group we may start with an arbitrary abelian group having either two or three independent generators in the restricted sense of this term. In the former case we may assume that an additional operator which transforms the independent generator whose order is at least as large as the order of the other into itself multiplied by this other which it transforms into its inverse. Hence it is commutative with the product of the square of the former independent generator into the latter and may be assumed to have this product for its square. To the identity of the quaternion quotient group there corresponds the group generated by the fourth power of the former, the square of the latter of the given independent generators, and the product of the latter into the square of the former.

If the arbitrary abelian group has three independent generators s_1, s_2, s_3 we may assume that the order of s_1 does not exceed that of s_2 and that the additional operator has s_3 for its square, transforms s_1 into its inverse and transforms s_2 into itself multiplied by s_1. The subgroup corresponding to the identity of the quaternion quotient group is then generated by the squares of s_1 and s_3, by the fourth power of s_2 and by the products of s_1 into s_2^2 and s_3. To prove that there exist an infinite number of such groups which have either the octic group or the abelian group of type (2,1) as a quotient group we may note that in each of these cases it may be assumed that G contains an arbitrary abelian subgroup having either two or three independent generators. The arguments in each of these cases are similar to those employed above when the quotient group is the quaternion group.

"RIEMANNIAN" DIFFERENTIAL GEOMETRY IN ABSTRACT SPACES[1]

BY A. D. MICHAL

DEPARTMENT OF MATHEMATICS, CALIFORNIA INSTITUTE OF TECHNOLOGY

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1. *Introduction.*—In this note brief indications are given of a set of postulates for "Riemannian" differential geometry in abstract spaces. The detailed treatment of this geometry has been included in a comprehensive memoir that I intend to publish elsewhere. An introduction to more general abstract differential geometries with one or several linear connections has been given by me in another comprehensive memoir.[2]

It has become clear in recent years that a differential geometry is concerned with *several* spaces together with their interconnections by means of interspace functional transformations. Moreover, the differentiability of at least one of these functional transformations enters into the theory in an essential manner. It is evident from these remarks that some of the spaces must be sufficiently restricted so as to be capable of supporting a theory of differentials.[3]

Let R be the class of real numbers, which we take as a known system.[4] The universe of discourse of our abstract Riemannian geometry consists of two classes E_1 and E_2 of undefined elements, two relations of equality,[5] two operations of addition, two operations of multiplication by elements of R, two norms, one interspace inner product $[.\ ,\ .]$, two metric functions, $g(.,.)$, $\bar{g}(.,.)$, two "parallelism" functions $g(.,.,.)$, $f(.,.,.)$ and two "curvature" functions $g(.,.,.,.)$, $f(.,.,.,.)$. The theory of this universe of discourse subject to the *sixty postulates* (classified into eight groups) with $[\delta x,\ g(x,\ \delta x)]^{1/2}$ as element of arc length constitutes our abstract Riemannian differential geometry. In the first comprehensive memoir referred to above we have stated the sixty postulates in such a manner as to make each postulate intelligible independently of the remaining ones. In order to avoid lengthy statements in the present note we have not done this for the thirty postulates that are explicitly given in section 2.

The classes E_1 and E_2 have been shown to have all the properties of a complete normed vector space.[6] The commutativity and associativity of each of the two additions, the usual three properties of each of the two equalities, the existence of the two zero elements and several continuity, differentiability and substitution properties are not postulated but proved as theorems.

Besides the classical finite dimensional examples and the author's infinitely dimensional instances,[7] several new instances and specializations of the general abstract theory have been found.

2. *The Postulates.*—

Group I: Postulates for E_1

Group II: Postulates for E_2

Group III: Postulates for interspace inner product $[\xi,\eta]$

(1) If $\xi\epsilon E_1$, $\eta\epsilon E_2$ then $[\xi, \eta]\epsilon R$; (2) $[\xi_1 + \xi_2, \eta] = [\xi_1, \eta] + [\xi_2, \eta]$; (3) $[\xi, \eta_1 + \eta_2] = [\xi, \eta_1] + [\xi, \eta_2]$; (4) $|[\xi, \eta]| \leq M\|\xi\|\|\eta\|$; (5) $[\xi, \eta] = 0$ for all ξ implies $\eta + \eta_1 = \eta_1$ for all η_1; (6) $[\xi, \eta] = 0$ for all η implies $\xi + \xi_1 = \xi_1$ for all ξ_1.

Group IV: Postulates for metric $g(x, \xi)$ and its inverse $\bar{g}(x, \eta)$

(1) If $x\epsilon E_1((x_0)_a)$, $\xi\epsilon E_1$, then $g(x, \xi)\epsilon E_2$; (2) $\|g(x, \xi)\| \leq M_1\|\xi\|$; (3) if $x_1 = x_2$, then $g(x_1, \xi) = g(x_2, \xi)$; (4) if $x\epsilon E_1((x_0)_a)$, $\eta\epsilon E_2$, then $\bar{g}(x, \eta)\epsilon E_1$; (5) if $x_1 = x_2$, then $\bar{g}(x_1, \eta) = \bar{g}(x_2, \eta)$; (6) $g(x, \bar{g}(x, \eta)) = \eta$; (7) $\bar{g}(x, g(x, \xi)) = \xi$; (8) $[\xi_1, g(x, \xi_2)] = [\xi_2, g(x, \xi_1)]$; (9) $[\xi, g(x, \xi)] \geq 0$; (10) if $[\xi, g(x, \xi)] = 0$, then $\xi + \xi_1 = \xi_1$ for all ξ_1.

Group V: First group of postulates for parallelism

(1) If $x\epsilon E_1((x_0)_a)$, ξ, $y\epsilon E_1$, then $g(x, \xi, y)\epsilon E_2$; (2) $g(x, \xi, y_1 + y_2) = g(x, \xi, y_1) + g(x, \xi, y_2)$; (3) given an $\epsilon > 0$ there exists a $\delta > 0$ such that $\|y\| < \delta$ implies $\|g(x, \xi, y)\| < \epsilon$; (4) given an $\epsilon > 0$ there exists a $\delta > 0$ such that $\|y\| < \delta$ implies $\|(g(x + y, \xi) + (-1)g(x, \xi)) + (-1)g(x, \xi, y)\| \leq \epsilon\|y\|$; (5) if $x_1 = x_2$, then $g(x_1, \xi, y) = g(x_2, \xi, y)$; (6) if $\xi_1 = \xi_2$, then $g(x, \xi_1, y) = g(x, \xi_2, y)$; (7) if $y_1 = y_2$, then $g(x, \xi, y_1) = g(x, \xi, y_2)$.

Group VI: Second group of postulates for parallelism

(1) If $x\epsilon E_1((x_0)_a)$, ξ_1, $\xi_2\epsilon E_1$, then $f(x, \xi_1, \xi_2)\epsilon E_2$; (2) $[y, f(x, \xi_1, \xi_2)] = [\xi_1, g(x, \xi_2, y)]$; (3) given an $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\xi_1\| < \delta$ implies $\|f(x, \xi_1, \xi_2)\| < \epsilon$.

Group VII: First group of postulates for curvature

(1) If $x\epsilon E_1((x_0)_a)$, ξ, y_1, $y_2\epsilon E_1$, then $g(x, \xi, y_1, y_2)\epsilon E_2$; ...; (4) given an $\epsilon > 0$ there exists a $\delta > 0$ such that $\|y_2\| < \delta$ implies $\|(g(x + y_2, \xi, y_1) + (-1)g(x, \xi, y_1)) + (-1)g(x, \xi, y_1, y_2)\| \leq \epsilon\|y_2\|$; ....

Group VIII: Second group of postulates for curvature

(1) If $x\epsilon E_1((x_0)_a)$, ξ_1, ξ_2, $y\epsilon E_1$, then $f(x, \xi_1, \xi_2, y)\epsilon E_2$; ...; (4) given an $\epsilon > 0$ there exists a $\delta > 0$ such that $\|y\| < \delta$ implies $\|(f(x + y, \xi_1, \xi_2) + (-1)f(x, \xi_1, \xi_2)) + (-1)f(x, \xi_1, \xi_2, y)\| \leq \epsilon\|y\|$; ....

3. *Parallelism and Curvature.*—Each function considered in this section has been shown to be continuous jointly in all its arguments and, except for the dependence on the variable x, linear in each argument. The differentials[8] $g(x, \xi; \delta x)$, $g(x, \xi; \delta_1 x; \delta_2 x)$, $f(x, \xi_1, \xi_2; \delta x)$ exist equal to $g(x, \xi, \delta x)$, $g(x, \xi, \delta_1 x, \delta_2 x)$, $f(x, \xi_1, \xi_2, \delta x)$, respectively.

The following two functions $\Gamma_1(x, \xi_1, \xi_2)$ and $\Gamma_2(x, \eta, \xi)$ play a rôle analogous to that of the Christoffel symbols of the second kind in n-dimensional classical Riemannian geometry. They are defined by

$$\begin{aligned}\Gamma_1(x, \xi_1, \xi_2) &\equiv {}^1/_2\, \bar{g}(x, g(x, \xi_1; \xi_2) + g(x, \xi_2; \xi_1) - f(x, \xi_1, \xi_2)) \\ \Gamma_2(x, \eta, \xi) &\equiv {}^1/_2 \left\{ g(x, \bar{g}(x, \eta); \xi) + f(x, \bar{g}(x, \eta), \xi) - g(x, \xi; \bar{g}(x, \eta)) \right\}\end{aligned} \tag{1}$$

and have been shown to satisfy the identity

$$[\Gamma_1(x, \xi_1, \xi_2), \eta] = [\xi_1, \Gamma_2(x, \eta, \xi_2)].$$

The "covariant" differential $g(x, \xi_1/\xi_2)$ defined by

$$g(x, \xi_1/\xi_2) \equiv g(x, \xi_1; \xi_2) - g(x, \Gamma_1(x, \xi_1, \xi_2)) - \Gamma_2(x, g(x, \xi_1), \xi_2)$$

vanishes. Moreover, the equation $g(x, \xi_1/\xi_2) = 0$ considered as a functional equation with the functions Γ_1 and Γ_2 as unknowns has the unique solution (1). With the aid of the functions Γ_1 and Γ_2 it is possible to develop a theory of parallel displacements and geodesics.

Let

$$\gamma_1(x, \xi_1, \xi_2) \equiv g(x, \Gamma_1(x, \xi_1, \xi_2)), \gamma_2(x, \xi_1, \xi_2) \equiv \Gamma_2(x, g(x, \xi_1), \xi_2)$$

then the function $R_1(x, \xi_1, \xi_2, \xi_3)$ defined as follows plays a rôle similar to the Riemann Christoffel curvature tensor in classical Riemannian geometry

$$\begin{aligned}R_1(x, \xi_1, \xi_2, \xi_3) \equiv {}^1/_2 \{ & g(x, \xi_2; \xi_1; \xi_3) + f(x, \xi_1, \xi_3; \xi_2) - f(x, \xi_1, \xi_2; \xi_3) \\ & - g(x, \xi_3; \xi_1; \xi_2) \} + \gamma_2(x, \bar{g}(x, \gamma_1(x, \xi_1, \xi_3)), \xi_2) \\ & - \gamma_2(x, \bar{g}(x, \gamma_1(x, \xi_1, \xi_2)), \xi_3).\end{aligned}$$

The following identities have been shown to hold for the curvature form R_1:

$$R_1(x, \xi_1, \xi_2, \xi_3) = -R_1(x, \xi_1, \xi_3, \xi_2), [\xi, R_1(x, \xi_1, \xi_2, \xi_3)] = -[\xi_1, R_1(x, \xi, \xi_2, \xi_3)], [\xi, R_1(x, \xi_1, \xi_2, \xi_3)] = [\xi_2, R_1(x, \xi_3, \xi, \xi_1)], R_1(x, \xi_1, \xi_2, \xi_3) + R_1(x, \xi_3, \xi_1, \xi_2) + R_1(x, \xi_2, \xi_3, \xi_1) = 0.$$

The addition of a few postulates has enabled us to study abstract Riemannian geometry with torsion while an evident modification of the postulates has led to a geometry with abstract Hermitian differential metric. As an interesting by-product we mention the fact that a slight modification of the first six groups of postulates alone suffices for an abstract dynamics with ${}^1/_2\,[dx/dt, g(x, dx/dt)]$ as kinetic energy.

[1] Presented at the 1933 Pasadena meeting of the American Math. Soc. The present drastically reduced set of postulates was presented at the 1935 Stanford meeting of the Amer. Math. Soc. Cf. *Bull. Am. Math. Soc.*, **39**, 879 (Nov., 1933), and **41**, 195 (March, 1935).

[2] Presented at the 1933 Pasadena meeting of the Am. Math. Soc. Cf. *Bull. Am. Math. Soc.*, **39**, 880 (Nov., 1933).

[3] Fréchet, M., *Ann. Sc. Ec. Norm. Sup.*, *t.* **62**, 293–323 (1925).

[4] A more general theory will result if in the place of R we employ an abstract normed ring.

[5] We use the symbols $=, +, \|\ldots\|, \ldots$ in more than one sense. No confusion need arise as the context makes clear the meaning of each such symbol. It is worth while to mention here that the relation of equality $=$ for E_1 as well as for E_2 is not an independent primitive idea; for, an equivalent set of postulates can be given in which the equality $=$ for E_1 as well as for E_2 is *defined* by means of the properties of the corresponding norm function $\|\ldots\|$.

[6] Banach, S., *Théorie des Opérations Linéaires*, chapter IV, Warsaw, 1932; see also Fréchet, M., *Espaces Abstraits*, Paris, 1928.

[7] Michal, A. D., "Affinely Connected Function Space Manifolds," *Am. Jour. Math.*, **50**, 473–517, especially 509–517 (1928); "Differential Geometries of Function Space," these PROCEEDINGS, **16**, 88–94 (Jan., 1930). See also Peterson, T. S., "The Analogue of Weyl's Conformal Curvature Tensor in a Michal Functional Geometry," *Annali di Mat.*, **13**, 55–62 (1934).

[8] The first partial differential with respect to x of a function $F(x, y_1, \ldots, y_n)$ is denoted by $F(x, y_1, \ldots, y_n; \delta x)$. All differentials are taken in the Fréchet sense.

ASYMPTOTIC REPRESENTATIONS OF CONFLUENT HYPERGEOMETRIC FUNCTIONS

BY E. FISHER

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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In a number of problems in quantum mechanics there is need for simple asymptotic approximations to the confluent hypergeometric function $M_{k,m}(z)$. In the following note a combination of the recurrence and differential properties of these functions has led to such expressions for the cases $|z| \gg |k|, 1$; $|m| \gg |k|, 1$ and $|z| \gg |m|, 1$. The notation and fundamental formulas used are from Whittaker and Watson, *Modern Analysis*.

Case I. $|z| \gg |k|, 1$.

If we let $F_{k,m}(z) = z^{-k}e^{z/2}W_{k,m}(z)$, the function F satisfies the two equations,

$$\frac{m^2 - (k - 1/2)^2}{z^2} F_{k-1,m} + \left(1 - \frac{2k}{z}\right) F_{k,m} = F_{k+1,m}. \qquad (1)$$

$$\frac{d}{dz} F_{k,m} = \frac{(k - 1/2)^2 - m^2}{z^2} F_{k-1,m}. \qquad (2)$$

From equation (1), neglecting $0(z^{-1})$, we obtain

$$0 = \frac{m^2}{z^2} F_{k,m} + F_{k+1,m} - F_{k+2} = \left(\delta + 1/2 + \frac{1}{2} D\right)\left(\delta + 1/2 - \frac{1}{2} D\right) F_{k,m}$$

where $\delta F_{k,m} = F_{k+1,m} - F_{k,m}$ and $D^2 = 1 + \frac{4m^2}{z^2}$. A general solution of this difference equation is $c(m, z)\left(\frac{1}{2} + \frac{1}{2}D\right)^k + d(m, z)\left(\frac{1}{2} - \frac{1}{2}D\right)^k$. However, since $F_{k,m} \xrightarrow[z\to\infty]{} 1$, and since, when $R(k) < 0$, $\left(\frac{1}{2} - \frac{1}{2}D\right)^k \xrightarrow[z\to\infty]{} \infty$, the solution representing F must be

$$F_{k,m}(z) = c(m, z)\left[\left(\frac{1}{2} + \frac{1}{2}D\right)^k + 0(z^{-1})\right]. \tag{3}$$

Because of the branch point in D and because of our use of $z = \infty$ as a boundary point, we shall require that $|m| < |z/2|$, although this requirement may be more stringent than is necessary.

Equation (3) may be written

$$(F_{k-1,m}/F_{k,m}) = \frac{2}{D+1}[1 + \epsilon_{k,m}(z)]; \ \epsilon = 0(z^{-1}). \tag{4}$$

Substituting back in equation (1), and neglecting $0(z^{-2})$, we have
$0 = (D-1)\epsilon_{k,m} - (4k/z) + (D+1)\epsilon_{k+1,m}$;

or if we set $z\epsilon_{k,m} = \nu_{k,m} - \frac{D+1}{D^2} + \frac{2k}{D}$,

we have $(D-1)\nu_{k,m} + (D+1)\nu_{k+1,m} = 0$, so that $\nu_{k,m} = c(m, z)[(1 - D)/(1 + D)]^k$. This term also is ruled out by the requirement that F be finite at $z = \infty$, so that the first approximation is

$$(F_{k-1,m}/F_{k,m}) = \frac{2}{D+1} + \frac{2}{z}\left[\frac{2k}{D(D+1)} - \frac{1}{D^2}\right] + 0(z^{-2}). \tag{5}$$

By using $\epsilon_{k,m}(z)$ as $0(z^{-1})$, we now see, we have taken $1 + \frac{4m^2}{z^2} = 0(1) \gg 0(z^{-1})$. Even so, if z is large enough, we shall still be able to cover most of the range $m = 0$ to $m = \left[\frac{z}{2}\right]$.

In a similar manner we can obtain a second approximation

$$\frac{F_{k-1,m}}{F_{k,m}} = \frac{2}{D+1} + \frac{2}{z}\left(\frac{2k}{D(D+1)} - \frac{1}{D^2}\right) + \frac{1}{z^2}\left\{\frac{1}{D} + \frac{1}{D^3} + \frac{5}{D^5} - \frac{1}{D+1} - \frac{1}{(D+1)^2} + 4k\left[\frac{1}{(D+1)^2} - \frac{1}{D^2} - \frac{2}{D^4}\right] + 4k^2\left[\frac{1}{D} + \frac{1}{D^3} - \frac{1}{D+1} - \frac{1}{(D+1)^2}\right]\right\} + 0(z^{-3}). \tag{6}$$

Turning now to equation (2), we write it

$$\ln[F_{k,m}(x)] = \left[m^2 - \left(k - \frac{1}{2}\right)^2\right]\int_{z=\infty}^{z=x} (F_{k-2,m}/F_{k,m})d(1/z)$$

and, by utilizing (6), we obtain

$$\frac{2im \ln[F_{k,m}(z)]}{m^2 - (k - 1/2)^2} = 2\left[1 + \left(\frac{k - 1/2}{m}\right)^2\right]\left(\varphi - \tan\frac{\varphi}{2}\right) + \frac{i}{m}\ln\left[\cos^{4k}\left(\frac{\varphi}{2}\right)\sec\varphi\right] + \frac{\tan\varphi}{m^2}\left(k\sec\varphi - \frac{5}{12}\tan^2\varphi - k^2 - \frac{1}{4}\right) \quad (7)$$

where $\varphi = \sin^{-1}(2im/z)$. A third approximation adds to this solution the terms

$$\frac{i}{m^3}\left\{(21 + 22k - 4k^2 + 4k^3)\ln\left[\cos\left(\frac{\varphi}{2}\right)\right] - (7 + 2k + k^2)\ln(\cos\varphi) + \frac{51 - 33k}{40} - \frac{1}{3}k^3 + \left(2k - \frac{3}{4}\right)\sec\varphi + \left(k^2 - k - \frac{11}{8}\right)\sec^2\varphi + \left(\frac{k^3}{3} - \frac{5}{2}k\right)\sec^3\varphi + \frac{27 + 10k - 16k^2}{16}\sec^4\varphi + \left(\frac{27}{10}k - \frac{1}{40}\right)\sec^5\varphi - \frac{13}{16}\sec^6\varphi\right\}.$$

To test this formula the special case of the Bessel function was computed. The above formulas give $\varphi = \sin^{-1}\frac{n}{x}$

$$J_n(x) = \sqrt{\frac{2}{\pi x}}(\sec\varphi)^{1/2 - 1/8n^2}\cos\left\{\frac{\pi}{2}(n + 1/2) - x + \frac{n^4 - 1/16}{n^3}\left(\tan\frac{\varphi}{2} - \varphi\right) + \frac{n^2 - 1/4}{8n^3}\left(\tan\varphi + \frac{5}{3}\tan^3\varphi\right)\right\}\exp\left\{\frac{n^2 - 1/4}{2n^4}\left[\ln\left(\cos^{21}\frac{\varphi}{2}\sec^7\varphi\right) + \frac{51}{40} - \frac{3}{4}\sec\varphi - \frac{11}{8}\sec^2\varphi + \frac{27}{16}\sec^4\varphi - \frac{1}{40}\sec^5\varphi - \frac{13}{16}\sec^6\varphi\right]\right\}. \quad (8)$$

The exponential factor is the improvement of the third approximation over the second. It can be neglected when x is large, or when n is considerably smaller than x. For instance, neglecting the exponential term, we obtain $J_{10}(20) = 0.186$, $J_{95}(100) = 0.140$, and from Hayashi's tables we see that $J_{10}(20) = 0.186$ and $J_{95}(100) = 0.137$. In the following cases the third approximation was needed, however: $J_{15}(20) = 0.004$, $J_{18}(20) =$

0.23, $J_5(10) = 0.234$, $J_8(10) = 0.29$. From Hayashi we obtain 0.001, 0.25, −0.234, 0.32, respectively.

Case II. $|m| \gg |k|$, 1.

The function $N_{k,m}(z) = z^{-1/2-m} M_{k,m}(z)$ has the following properties

$$\frac{k^2 - (m + 1/2)^2}{(2m+1)^2(2m+2)} z^2 N_{k,m+1} + \left[\frac{4mkz}{4m^2 - 1} - 2m\right] N_{k,m} + 2mN_{k,m-1} = 0 \; N_{k,m}(z) \underset{m\to\infty}{\longrightarrow} 1. \tag{9}$$

If $N_{k,m}(z) = \left(\frac{4m}{ze}\right)^m Q_{k,m}(z)$, then, neglecting $0(m^{-1})$, equation (9) becomes

$$0 = Q_{m+2} + \frac{4m}{z} Q_{m+1} - Q_m = \left[\delta^2 + \left(2 + \frac{4m}{z}\right)\delta + \frac{4m}{z}\right] Q_m.$$

$$= \left(\delta + 1 + \frac{2m}{z} + \sqrt{1 + \frac{4m^2}{z^2}}\right)\left(\delta + 1 + \frac{2m}{z} - \sqrt{1 + \frac{4m^2}{z^2}}\right) Q_m.$$

The last step is valid provided we require that $\sqrt{1 + \frac{z^2}{4m^2}} \gg 0 \; (m^{-1})$.

Of the two solutions of this difference equation, one has an infinity at $m = \infty$, and so the first approximation reduces to

$$(N_{k,m+1}/N_{k,m}) = \frac{8m^2}{z^2}(E - 1)\,[(1 + \epsilon_{km}(z)]; \qquad E^2 = 1 + \frac{z^2}{4m^2}$$
$$\epsilon = 0(m^{-1}). \tag{10}$$

Substituting this back in equation (9) and neglecting $0(m^{-2})$, we obtain a second approximation, by methods similar to those for case I,

$$\frac{N_{k,m+1}}{N_{k,m}} = \frac{8m^2}{z^2}(E-1)\left[1 + \frac{1}{2m}\left(2 + \frac{kz}{mE} - \frac{1}{E} - \frac{1}{E^2}\right) + 0\,(m^{-2})\right]. \tag{11}$$

To solve this, we notice that if

$$N_{k,m} = \left(\frac{8m^2}{z^2}\right)^m (E-1)^m e^{Kk,m}(z), \text{ then}$$

$$K_{k,m+1} - K_{k,m} = -1 + \frac{1}{E} + \frac{1}{2m}\left(1 + \frac{kz}{mE} + \frac{1}{E} - \frac{1}{E^2} - \frac{1}{E^3}\right) + 0(m^{-2}).$$

Neglecting $0(m^{-2})$, the solution of this is

$$K_{k,m} = mE - m + k \ln\left(E - \frac{z}{2m}\right) - \frac{1}{2}\ln E + \psi\,(k, z).$$

Since $N_{km} \underset{m\to\infty}{\longrightarrow} 1$, it turns out that $e^{\psi} = 1$, so that the first approximation

for N is

$$N_{k,m}(z) = \left(\frac{8m^2}{z^2}\right)^m (E-1)^m \left(E - \frac{z}{2m}\right)^k E^{-1/2} e^{m(E-1)}. \quad (12)$$

In the same manner, by neglecting only $0(m^{-3})$, the second approximation equals the above solution multiplied by the factor

$$\exp\left[\frac{1}{24m}\left(2 - 12k^2 + \frac{12k^2+3}{E} + \frac{6kz/m}{E^2} - \frac{5}{E^3}\right)\right].$$

Since we have assumed that $E \gg 0(m^{-1})$, and have used $m = \infty$ as a boundary point, the above solution is perhaps valid only for $|m| > \left|\frac{z}{2}\right|$.

The above formulas give for the Bessel function

$$J_n(x) = \left(\frac{n^2}{x}\right)^n \frac{(1-u)^n}{\gamma(n+1)u^{1/2}} \exp\left[n(u-1) + \frac{1}{24n}\left(2 + \frac{3}{u} - \frac{5}{u^3}\right)\right] \quad (13)$$

where $u^2 = 1 - \frac{x^2}{n^2}$. With this formula we compute $J_{24}(20) = 0.0198$, where Hayashi's tables give 0.0199.

Case III. $|z| \gg |m|, 1$.

By the methods of case I we obtain for a first approximation:

$$F_{k,m}(z) = (2k)^{2k} e^{-k} \left[z\left(1 - \sqrt{1 - \frac{4k}{z}}\right)\right]^{-2k} \left(\frac{z}{z-4k}\right)^{1/4} \exp\left[\frac{z}{2}\left(1 - \sqrt{1 - \frac{4k}{z}}\right)\right] \quad (14)$$

provided $z \gg m, 1, R\left(\frac{1}{4} - \frac{k}{z}\right) > 0$ and $|k|, |k-1|, \left|\frac{1}{4} - \frac{k}{z}\right| \gg 0(z^{-1})$.

For computation a second or third approximation might be necessary. These would contain m.

I am indebted to Professor P. M. Morse for the interest he has shown in my work.

COMPLETELY INTEGRABLE DIFFERENTIAL EQUATIONS IN ABSTRACT SPACES[1]

BY A. D. MICHAL AND V. ELCONIN

DEPARTMENT OF MATHEMATICS, CALIFORNIA INSTITUTE OF TECHNOLOGY

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1. *Introduction.*—This note briefly summarizes some of our results in the theory of the differential equation

$$d_\xi^x f(x) = F(x, f(x), \xi), \tag{1}$$

where the left member is the Fréchet differential[2] of $f(x)$ with increment ξ, and the ranges and domains of the functions involved are in Banach spaces.[3] The rather intricate proofs, which involve most of the known and many new properties of abstract differentials and integrals,[4] are too long to be given here; but they are all included in a comprehensive memoir which we intend to publish elsewhere. A few indications are given of the wide range of results obtainable either by specializing the Banach spaces, or by applying the general theory to other problems in abstract analysis.

2. *Fundamental Existence Theorems.*—In this section E, Σ are Banach spaces; x, u, ξ, z are in E; y, v, η are in Σ; a, b, c, g are positive real numbers; m is a non-negative integer; and $\phi(x)$ is any function on E to Σ.

THEOREM I. *For any x, y, z, ξ, η such that $\|x - u\| < a$ and $\|y - v\| \leq b$, let $F(x, y, z)$ be in Σ, linear[5] in z, and such that*

$$G(x, y, z, \xi, \eta) \equiv d_\eta^y d_\xi^x F(x, y, z)$$

exists, continuous in x, y; and for $\|x - u\| < a$ and any m, let $f_0(x)$ be v, and let $f_{m+1}(x)$ be

$$v + \int_0^1 F(u + \sigma(x - u), f_m(u + \sigma(x - u)), x - u)d\sigma.$$

Then

(1) *For some $c \leq a$ and any x such that $\|x - u\| < c$, $f_m(x)$ exists and $\|f_m(x) - v\| < b$.*

(2) *For any such c: if*

$$\|G(x, y, z, 0, \eta)\| \leq g\|z\|\|\eta\|$$

for some g and any x, y, z, η such that $\|x - u\| < c$ and $\|y - v\| \leq b$, then

(i) $f(x) \equiv \lim_{m \to \infty} f_m(x)$ *exists for $\|x - u\| < c$;*

(ii) $$G(x, y, z, \xi, F(x, y, \xi)) = G(x, y, \xi, z, F(x, y, z)) \tag{2}$$

for any x, y, z, ξ *such that* $\|x - u\| < c$ *and* $\|y - v\| \leq b$ *implies*[6] $f(u) = v$, $\|f(x) - v\| \leq b$, $d_\xi^x f(x) = F(x, f, \xi)$, $d_z^x d_\xi^x f(x) = G(x, f, \xi, z, F(x, f, z))$ *for any* x, ξ, z *such that* $\|x - u\| < c$;

(iii) if $\phi(u) = v$, $\|\phi(x) - v\| \leq b$, $d_\xi^x \phi(x) = F(x, \phi, \xi)$,

for any x, ξ *such that* $\|x - u\| < c$, *then for* $\|x - u\| < c$, $\phi(x) = f(x)$.

This theorem shows the existence, under certain hypotheses on $F(x, y, z)$, of a unique solution $f(x)$ of equation (1) *after* u and v have been fixed; but it can be shown that under similar hypotheses a unique solution $f(x, v, u)$ exists for all u, v in certain domains of E and Σ, respectively. This last theorem, to which we shall refer as Theorem II, enables us to study the dependence of the solution on the initial parameters u, v. We have found that $f(x, v, u)$ is continuous in v and that under certain additional hypotheses on $F(x, y, z)$, the Fréchet differentials of $f(x, v, u)$ with respect to u and v exist. However, it appears from our latest work that these additional hypotheses can be weakened considerably.

3. *Specializations and Applications.*—Many known results may be obtained from the theorems of the last section by properly specializing E or Σ. Thus if E is the real number system, then equation (1) is equivalent to the "ordinary" differential equation

$$\frac{df(x)}{dx} = \phi(x, f(x)),$$

where $\phi(x, y)z = F(x, y, z)$, the integrability condition is automatically satisfied, and Theorem I includes a theorem of Kerner.[4] Or, if E is the space of all ordered sets $(x_1, \ldots, x_m)$ of m real numbers, and Σ is the space of all ordered sets $(y_1, \ldots, y_n)$ of n real numbers, then equation (1) is equivalent to the system of partial differential equations

$$\frac{\partial}{\partial x_k} f_i(x_1, \ldots, x_m) = \phi_{ik}(x_1, \ldots, x_m; f_1, \ldots, f_n) \quad i = 1, 2, \ldots, n; \quad k = 1, 2, \ldots, m,$$

on putting $F(x, y, z)$ equal to

$$\left(\sum_1^m{}_k \phi_{ik}(x_1 \ldots x_m; y_1 \ldots y_n)\right)z_k, \ldots, \sum_1^m{}_k \phi_{nk}(x_1 \ldots x_m; y_1 \ldots y_n)z_k),$$

and Theorems I and II include theorems of Nikliborc[7] and Nikodym,[8] respectively.

New results may also be obtained in this way. For example, if E is the space of real, continuous functions $x(s)$ on a real interval (α, β), and Σ is the real number system, then the functional equation[9]

$$d_\xi^x f[\overset{\beta}{x}(s)] = \int_\alpha^\beta \phi[\overset{\beta}{x}(s)/f[\overset{\beta}{x}(s)], t]\xi(t)dt \tag{3}$$

is an instance of equation (1), on putting

$$F(x, y, z) = \int_\alpha^\beta \phi[\overset{\beta}{x}(s)/y, t]z(t)dt;$$

and Theorems I and II become the strongest existence theorems known to us. In fact, they furnish existence theorems for functional equations much more complicated than equation (3).

The theory of the differential equation (1) yields an existence theorem, under certain hypotheses on $K(x, y)$, for the "implicit" function $I(x)$ satisfying

$$K(x, I(x)) = 0$$

where the ranges and domains of $I(x)$ and $K(x, y)$ are in Banach spaces. In making this application, we found it necessary to show that the inverse of a solvable[10] linear function Fréchet differentiable with respect to a parameter has itself a Fréchet differential with respect to that parameter; a result of considerable intrinsic interest.

Another application of the theory is to the second order differential equation

$$d_\xi^x d_\eta^x f(x) = H(x, f(x), d_\eta^x f(x), \xi, \eta).$$

Finally, we emphasize again that only a few of the problems in which the theory of equation (1) is useful have been indicated here. More complete indications will be given in other papers.

[1] Presented to the Am. Math. Soc. at the 1934 Berkeley meeting and at the 1934 Los Angeles meeting. Cf. *Bull. Am. Math. Soc.*, **40**, 530 and 814 (1934).

[2] Fréchet, M., *Annales Sc. Ec. Norm. Sup., t.* **42**, 293–323 (1925). By $d_{\xi_1 \ldots \xi_n}^{x_1 \ldots x_n} f(x^1, \ldots, x_n)$ is meant the Fréchet differential of $f(x_1, \ldots, x_n)$ with increments $\xi_1, \ldots, \xi_n$.

[3] Banach, S., *Fund. Math.*, **3**, 133–181 (1922). See also his book, *Théorie des Opérations Lineaires* (1932). A Banach space is briefly a complete normed vector space with real number multipliers. We shall assume that if $\alpha = \beta$, then α may be replaced by β in any proposition.

[4] Kerner, M., *Prace Matematyczno-Fizyczne*, **40**, 47–67 (1932). By $\int^b f(\sigma)d\sigma$ is meant the "Riemann integral" of the abstract function $f(\sigma)$ of a real variable σ.

[5] A function is linear if it is additive and continuous.

[6] Identity (2) is the condition of complete integrability for equation (1).

[7] Nikliborc, W., *Studia Math.*, **1**, 41–49 (1929).

[8] Nikodym, O., *Bull. Internat. De L'Academie Polonaise Sci. Lets.* No. 4–8, 134–144 (1933).

[9] For the functional notation see Volterra, V., *Functions de Lignes*, Paris, 1913.

[10] Hildebrant, T. H., and Graves, L., *Trans. Amer. Math. Soc.*, **29**, 145 (1927). These authors use the phrase "linear function with a reciprocal" instead of "solvable linear function."

A REFINEMENT OF THE HEISENBERG THEORY OF FERROMAGNETISM, APPLICABLE TO SIMPLE CUBIC CRYSTALS

BY CHARLES H. FAY

RESEARCH LABORATORY OF PHYSICS, HARVARD UNIVERSITY

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Introduction.—The Heisenberg theory of ferromagnetism,[1] in what we shall term its "simple" form,[2] involves the replacement of the energy levels of the spin multiplets of the micro-crystal for a given total spin quantum number S' by the average energy for S'. As a supposedly better approximation Heisenberg has used the assumption of a Gaussian distribution of energies for S'; however, this approximation is in some respects worse than the simple approximation, for it has the effect of introducing energies lower than that of the state of maximum S' for states of small spin. Hence, even though ferro magnetism may be obtained in a certain temperature range, the crystal becomes paramagnetic at very low temperatures, instead of approaching its true saturation magnetization as it should. The work of the present paper was undertaken largely in order to get a better idea of the validity of the simple approximation described above.

Method.—The method of the present paper consists essentially of regarding the micro-crystal as built up of units larger than the atom but containing sufficiently few atoms to allow calculation of their unperturbed energy levels; the coupling energies between these units for states of spin S' are replaced by the average coupling energy for S'. The method is most readily applicable to simple cubic crystals, which may be regarded as built up of unit cubes each containing eight atoms. In restricting ourselves to simple cubic crystals we are unfortunately giving up the chance of comparing our results with those of experiment, as simple cubic arrangement of atoms considered ferromagnetic is found only in certain little-studied alloys. Our interest, however, lies in the comparison of the results of this method with those of the simple Heisenberg theory. It will be seen that our method treats half of the interaction rigorously (instead of none as previously), handling the remainder by an average energy approximation similar to that of the simple theory. This retention of some of the fine structure of the problem will give us a better idea of the results to be expected were a rigorous treatment possible.

The energy levels of the problem of eight electrons in similar orbits at the corners of a cube have been worked out by Seitz and Sherman[3] and Serber.[4] In order to be able to use their results we further restrict ourselves to the case of atomic spin 1/2.

It is assumed as usual that only the exchange integral between orbits of neighboring atoms is appreciable. Now of the six neighbors of a particu-

lar atom three are in its unit cube, so that half of the atomic coupling is taken care of in the cube levels. The average intercube energy for total spin S' is thus half the average interatomic energy for S', being $(1/2)$ $(-2\alpha\Sigma\mathbf{s}_i\cdot\mathbf{s}_j)$, apart from an additive term independent of spin orientation.[5] The summation extends over all pairs of neighboring atoms. Here α is the Heisenberg exchange integral between valence orbits of neighboring atoms; $\mathbf{s}_i$ is the vector spin matrix of atom i. We shall here use n for the number of cubes, so that the number of atoms is $8n$. Now

$$(\sum\mathbf{s}_i)^2 = \sum\mathbf{s}_i^2 + 2\sum_{i<j}\mathbf{s}_i\cdot\mathbf{s}_j = S'(S'+1),$$

whence

$$\overline{\mathbf{s}_i\cdot\mathbf{s}_j} = S'(S'+1)/8n(8n-1) - 3/4\ (8n-1),$$

so that to significant terms

$$\sum\overline{\mathbf{s}_i\cdot\mathbf{s}_j} = 3S'^2/8n. \qquad (1)$$

The summation in (1) is over neighboring atoms. To our approximation the portion of the energy depending on the spin is thus

$$W_{S'} = \sum_\mu W_\mu - 3\alpha S'^2/8n, \qquad (2)$$

where W_μ is the unperturbed energy of cube μ.

The partition function for the system in a magnetic field H along the z axis is from (2)

$$Z = \sum[\prod_\mu \exp(-W_\mu/kT)]\exp(3\alpha S'^2/8nkT - 2\beta HS_z'/kT), \qquad (3)$$

the summation being over the states of the micro-crystal. Here β is the value of the Bohr magneton. The coefficient of H arises from the usual assumption that only the spin angular momentum need be considered in determining the magnetic moment.

In order to sum (3) we make use of the assumption, true of the other theories, that the important terms in Z lie in the neighborhood of $S' = -S_z' = -S_z^\circ$, so that we may replace S'^2 by $-S_z^{\circ 2} + 2S_z^\circ S_z'$ in summing (3). This procedure gives, since $S_z' = \sum_\mu S_z^{(\mu)}$,

$$Z = \exp(-3\alpha S_z^{\circ 2}/8nkT)[\sum{}'\exp(-W_\mu/kT + (3\alpha S_z^\circ/4nkT - 2\beta H/kT)S_z^{(\mu)})]^n, \qquad (4)$$

the summation in (4) being over the states of the cube. It is convenient to introduce the notation

$$\zeta = -S_z^\circ/4n, \qquad \gamma = 3\alpha/kT, \qquad y = \gamma\zeta + 2\beta H/kT, \qquad (5)$$

ζ being the ratio of the magnetic moment to its true saturation value at $T = 0$, and

$$\{x\} = \sum_x \exp(-W(\text{cube})/kT). \tag{6}$$

Σ_x denotes a summation over cube states of spin x, not including the spatial degeneracy factor. Since the cube levels are proportional to α, the quantities (6) are functions of γ only. Equation (4) may now be written

$$\begin{aligned} Z = \exp(-2n\gamma\zeta^2)[2\{4\}\cosh 4y + 2(\{4\} + \{3\})\cosh 3y + \\ + 2(\{4\} + \{3\} + \{2\})\cosh 2y + 2(\{4\} + \{3\} + \{2\} + \\ + \{1\})\cosh y + (\{4\} + \{3\} + \{2\} + \{1\} + \{0\})]^n. \end{aligned} \tag{7}$$

From the assumption made in deriving (4) it follows that

$$\zeta = (1/4n)\partial \ln Z/\partial y. \tag{8}$$

From (7) and (8) we have finally

$$\begin{aligned} \zeta = [8\{4\}\sinh 4y + 6(\{4\} + \{3\})\sinh 3y + 4(\{4\} + \\ + \{3\} + \{2\})\sinh 2y + 2(\{4\} + \{3\} + \{2\} + \\ + \{1\})\sinh y]/4[2\{4\}\cosh 4y + 2(\{4\} + \{3\})\cosh 3y + \\ + \ldots\ldots\ldots + (\{4\} + \{3\} + \ldots\ldots\ldots + \{0\})], \end{aligned} \tag{9}$$

which has besides the trivial root zero for $H = 0$ a real positive root for sufficiently large γ. The ferromagnetic Curie temperature, defined as the temperature at which the pseudo-saturation magnetism vanishes, is thus determined by the value of γ for which (9) has the double root zero. This condition gives

$$\gamma_c\varphi(\gamma_c) = 1, \tag{10}$$

where

$$\begin{aligned} \varphi(\gamma) = (30\{4\} + 14\{3\} + 5\{2\} + \{1\})/(18\{4\} + \\ + 14\{3\} + 10\{2\} + 6\{1\} + 2\{0\}). \end{aligned} \tag{11}$$

Results below Curie Temperature.—Equation (10) has been solved numerically with the result $\gamma_c = 1.1795$, whereas the "simple" Heisenberg theory gives the value $\gamma_c = 1$. The Curie temperature as given by our method is thus 0.8478 times that of the simple Heisenberg theory. That the results are so little different is of considerable interest in view of the fact that the Gaussian distribution approximation does not admit ferromagnetism for simple cubic crystals.

The pseudo-saturation magnetization curve has been obtained by numerical solution of (9). Figure 1 gives this curve, plotted against the reduced temperature T/T_c. For purposes of comparison the pseudo-saturation magnetization curve given by the simple theory is also plotted against the appropriate reduced temperature. It is seen that on this basis the

curves differ appreciably only at fairly low temperatures; the success of applications of the theory using the experimental value of the Curie temperature to determine the value of the exchange integral is thus seen to afford no very decisive check on the validity of the approximations used. At very low temperatures (9) becomes approximately

$$\zeta = 1 - (1/4) \exp(-3\alpha/kT), \tag{12}$$

while the simple theory gives

$$\zeta = 1 - 2 \exp(-6\alpha/kT). \tag{13}$$

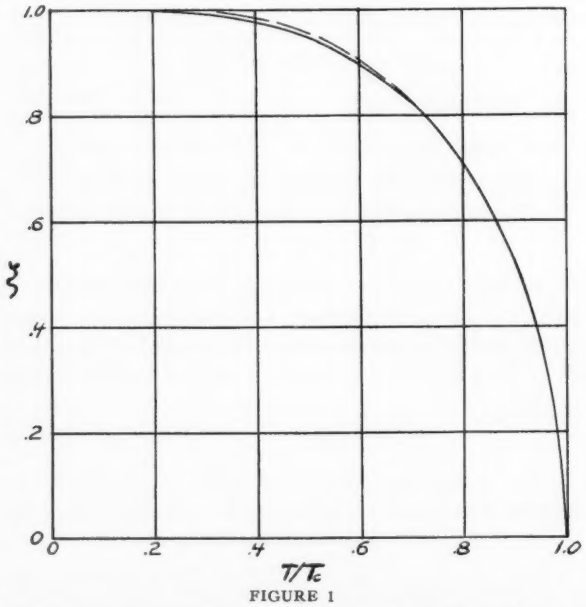

FIGURE 1

Psuedo-saturation magnetization curves. The dashed Curie indicates the result of the simple Heisenberg theory.

A theory for extremely low temperatures due to Bloch[6] gives

$$\zeta = 1 - 0.0587(kT/\alpha)^{3/2}. \tag{14}$$

While our result retains the exponential character of (13) its trend as compared to (13) is toward the slower approach to saturation given by (14).

Behavior Above Curie Point.—For temperatures above the Curie point the

susceptibility for small fields may be obtained from (9) by retaining only first order terms in y. The result is

$$\chi = 2N\beta^2\varphi(\gamma)/k(T - \gamma_c\varphi(\gamma)T_c). \quad (15)$$

In figure 2 (15) is compared with the result of the simple Heisenberg theory,

$$\chi = N\beta^2/k(T - T_c), \quad (16)$$

the reciprocal susceptibilities being plotted, as is customary. Experimentally it is found that the susceptibility is given to considerable accuracy

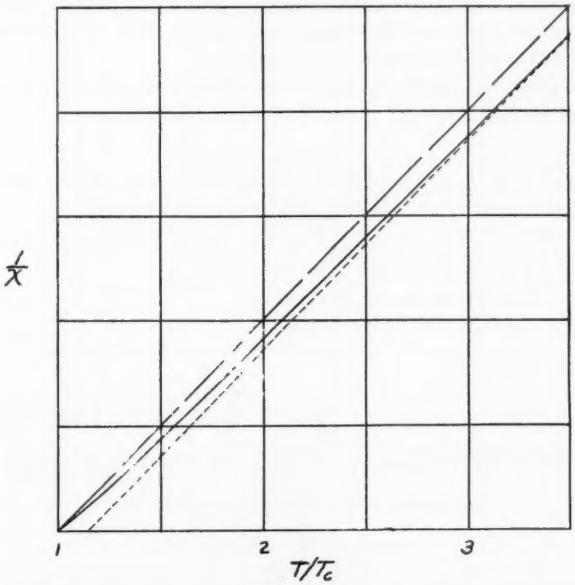

FIGURE 2

Reciprocal susceptibility curves above the Curie point. *Dashed line*, result of simple theory; *solid curve*, our result; *dotted line*, its asymptote.

by a modified Curie law like (16), with, however, a T_c different from the ferromagnetic Curie point, defined as the second or paramagnetic Curie point. Now (15) gives a curve which approaches rather gradually an asymptote with intercept at $T = 1.14T_c$, some 90° above the ferromagnetic T_c for T_c of the order of 640°K. (nickel). This value is hardly to be compared with experimental values for the difference between the Curie points, for the temperature range covered by experiment does not much exceed $2T_c$. A tangent to our curve in the experimental range would give a value

of the order of 10° for the difference between the two Curie points, supposing that the second Curie point could be qualitatively explained on this sort of basis. The experimental value for the common magnetic materials is about 20°. However, a paper by Forrer[7] seems to show that hysteresis is required to account for the existence of two Curie points.

Conclusion.—The fact that our method, which takes account of one-half the interaction rigorously, gives results comparing closely with those of the simple Heisenberg theory, but differing widely from the result of the use of the Gaussian distribution, seems to indicate that the simple approximation is better than the Gaussian one. The order of validity of the simple approximation is perhaps as good as that of the other assumptions of the Heisenberg theory, except at low temperatures.

In conclusion the author wishes to thank Professor J. H. Van Vleck for suggesting the problem and for many helpful discussions.

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ELECTRICAL CONDUCTANCE OF SHORT GAPS IN AIR

BY E. HUTCHISSON, T. H. OSGOOD* AND R. E. FEARON

UNIVERSITY OF PITTSBURGH, PITTSBURGH, PA.

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The transition between an open circuit and electrical contact is generally considered to be abrupt. Yet it seems reasonable that if a gap could be closed gradually enough, there would be a continuous change from open circuit to good contact. Conduction through short gaps usually has been investigated[1] for cases in which the width of the gap is of the order of several wave-lengths of visible light. Very great potential differences are needed to produce measurable currents under such conditions, making it essential to use high vacua for the experiments, as otherwise gaseous discharges occur. It is the purpose of this work to study the dependence of conductance upon gap width under ordinary laboratory conditions (such as obtain for switch contacts in air).[2] To prevent sparking at high field intensities, potential differences are used which are below the ionizing potential of the common gases of air. The use of such small potentials makes it

essential to use exceptionally short gaps if the currents obtained are to be capable of measurement, even with modern direct current amplifiers.

In this research the gap considered is effectively that between a sphere and a plane. Assuming that the conductance falls off very rapidly as the gap width increases it can be shown by calculation that the conductances of very short gaps of this type of equal minimum width are proportional to

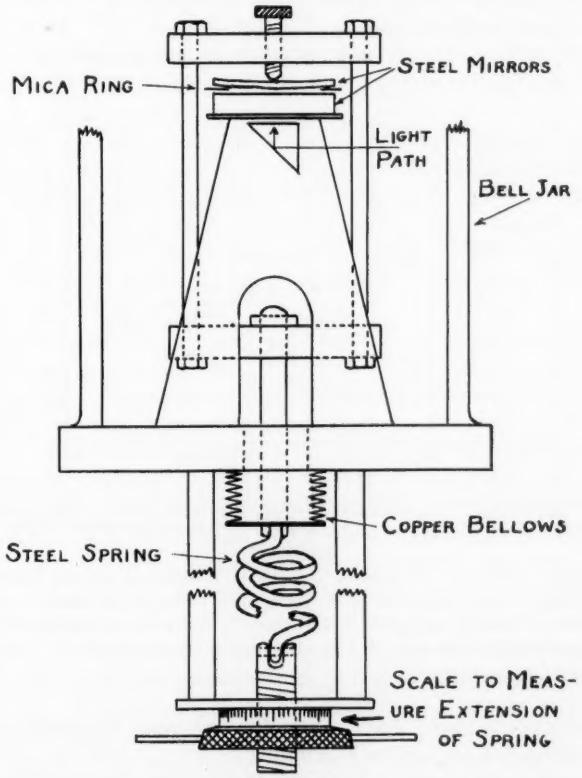

FIGURE 1

Apparatus used to bring two nearly flat surfaces into gradual contact.

the radii of curvature of the spherical surfaces. Three experimental arrangements were used in which the radii of curvature of the spherical electrodes were approximately 14×10^2 cm., 14 cm. and 25×10^{-4} cm. Although the surfaces used in the present work were not rigorously dust-free, all ordinary precautions to remove dust were taken, and it is felt that the results represent what may be expected to occur when ostensibly clean

metallic surfaces are brought into contact under everyday laboratory conditions.

One of the experimental arrangements consisted of a gap between a circular stainless steel mirror (plane to within $^1/_{10}$ wave-length of sodium light and 6 mm. thick) and a thin stainless steel mirror (polished plane to within one wave-length and 2 mm. thick); the latter being loaded at the center. The thin disc was supported above the other by a very thin ring of mica as shown in figure 1. The mirrors were 4.0 cm. in diameter and the hole in the mica ring was approximately 3.2 cm. in diameter. The load on the upper disc was applied by means of a stretched spring to avoid the

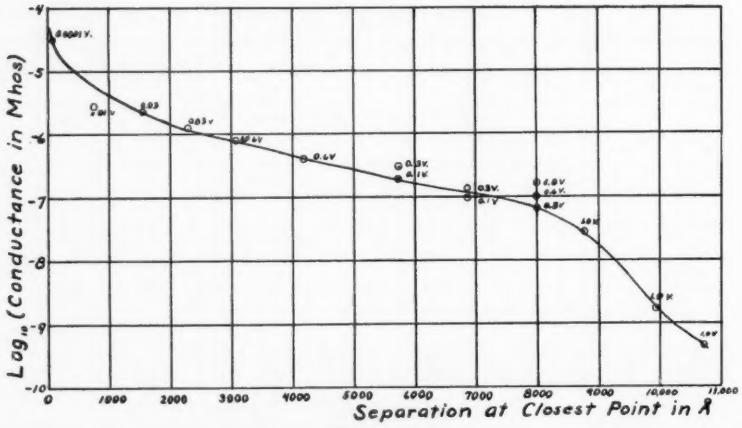

FIGURE 2

Conductance of an air gap between a plane and a sphere of 1430 cm. radius. Since the conductance varies somewhat with the applied potential difference, the latter is indicated at the right of each point. The curve is drawn approximately through the points representing the conductance at the smallest potential difference which would yield accurate measurements.

seismic effects of a direct load. Provision was made for covering the apparatus with a bell jar and for evacuating.

The bending of the upper disc was calibrated by observing interference fringes produced when the lower steel disc was replaced by a heavy piece of optical glass.

It was found that with a separating mica ring thin enough to show interference colors, the two plates acted as though they were in contact. With larger thicknesses of mica, contact occurred only when the upper disc was loaded. As contact was approached by loading, the conductance of the gap gradually increased. A fairly typical graph of the variation of conductance with distance of minimum separation is shown in figure 2.

The separation is considered to be zero when the conductance of the gap no longer changes appreciably with increasing load. Since conductance varies somewhat with applied potential difference, the potential difference for each point is recorded. A curve is drawn through points for which the applied potential difference was just sufficient to allow current measurements to be made. In its essential features, such a curve could be duplicated at will. The exact location of points changed somewhat upon prolonged application of the potential difference or even upon standing. Very much the same conductance values were obtained when a bell jar was placed over the plates and a vacuum of approximately 0.01 mm. of mercury maintained for several days.

Tests for the validity of Ohm's law were made at various separations. For large separations the current is approximately proportional to the square of the voltage. At about 8000 Å where the change of slope of the curve in figure 2 occurs, there is a region in which the current is nearly proportional to the three-halves power of the voltage. Below approximately 2000 Å, Ohm's law holds to within the accuracy of the measurements (about 5%).

An attempt was made after two years to repeat these measurements using the same steel mirrors. In the meantime several small scratches had developed. In general a curve of the same shape was obtained except that the points were very much more irregular. The change in minimum separation from the point at which the current just became perceptible to that of contact was only about one-fourth as large as in the earlier work.

A second set of observations was made using an entirely different type of apparatus in which the radius of curvature of the spherical electrode was about 14 cm. In this apparatus, by means of a compound lever a highly polished steel spherical electrode could be brought very gradually into contact with a stainless steel mirror (plane within a few wave-lengths). One division on the adjusting dial corresponded to a change in separation of 135 Å. The plane steel mirror was supported at three points and could be replaced by a plate of optical glass. The movement of the spherical electrode could therefore be calibrated by means of interference fringes. Electrical measurements were made in the same manner as in the loaded disc experiment and contact was assumed when further turning of the dial produced no further increase in current at a constant very small potential difference. In addition to the conductance measurements, it was possible to make *a direct comparison* of the position of optical contact and the position of electrical contact. These occur within 5000 Å of one another. Leaving the dial of the lever system at the same setting, it was possible to lift the steel mirror from its three-point support and replace it so that a new point on the same plane mirror was used to form the gap. The surfaces were always wiped off very carefully with lens paper before running a con-

duction experiment. The drawing of a narrow strip of lens paper between the surfaces when they are nearly in contact was tried, but it did not seem to have any effect on the results.

By replacing the steel plate by an optical flat at the close of a conduction

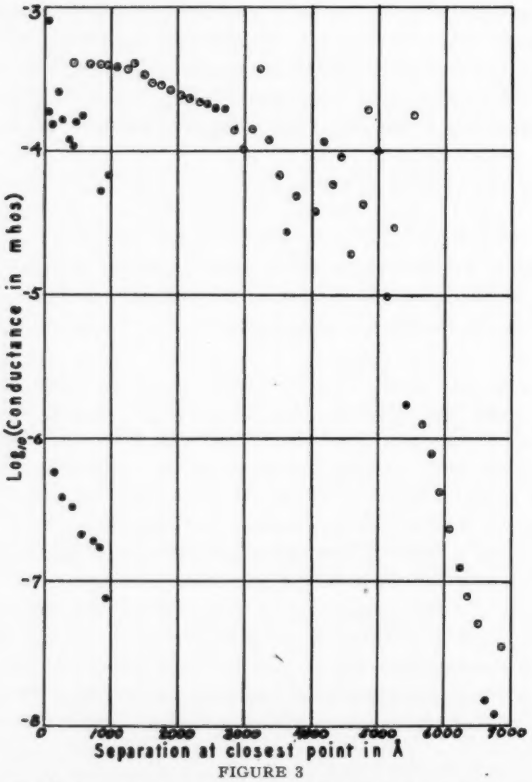

FIGURE 3

Conductance measurements of an air gap between a plane and a sphere of 14 cm. radius. The erratic series of points (circles with dots) was obtained with a newly prepared surface. Subsequent measurements with same surfaces yielded points indicated by circles with crosses in them and by the filled-in circles.

experiment, the region on the spherical electrode where contact occurred could be located by the interference method. Thus a minute inspection is possible of the region where the electric forces were the strongest. Such inspection revealed that the surface was not the same after the experiment

as it was at the beginning of the experiment. In each instance there were disturbed places, visible as darker areas in the interference pattern, approximately two microns wide and four or five microns long. When a spot on the spherical electrode was thus used up, a fresh spot could be used in the next experiment by tilting the three-point support and thus changing the point where the spherical surface was tangent to the plane.

Representative points showing the dependence of conductance upon gap width in this experiment are shown in figure 3. The long and extremely erratic series of points illustrates what was usually obtained when a newly prepared surface was used under scrupulously clean conditions. As the mirror was used, the repetition of the experiment gave successively more abruptly rising curves plotted between conductance and gap width. The more abrupt series of points were sometimes fairly reproducible, but the long, erratic ones never were. The change in the nature of the results as the mirrors were used may have been due to the production of sharp prominences on the surfaces by the destructive action of the electrical and mechanical factors which affected them. It is to be remembered in interpreting these points that the gap width was measured from some arbitrarily chosen point where the electrical conduction became very good. Thus, it may be that the gap widths quoted in figure 3 represent, not the separation of the general level of the surfaces, but the distances between conducting projections on either or both surfaces.

In the experiments using the lever, there were many abrupt changes in conductance as the separation was changed. This is to be ascribed to the breaking down of conducting bridges, possibly from mechanical causes and possibly from being burned up by the heating effect of currents that passed through them.

The electrical properties of a gap consisting of two wires of equal diameter crossed at right angles are the same as those of a gap between a plane and a spherical electrode of the same radius as that of the wires Fine tungsten wires were mounted on the opposite sides of the slow motion end of the lever system described in the previous paragraphs. The wires were stretched between supports 0.1 cm. apart. An estimate of the forces exerted transversely to the wires by the largest fields used is 0.1 dyne. This force would be sufficient to cause a displacement of the wire of not more than 25 Ångström units, assuming it is under a tension of 100 gram weights. The diameter of the wires used was 0.002 in.

The small radius of curvature of the electrodes in this experiment reduced very greatly the area over which the surfaces were very close as they approached contact. This change practically eliminated the averaging of the effect of foreign particles scattered over the surface. The result was that the system passed from an open circuit condition to electrical contact in a very abrupt manner. In fact, this change occurred in a distance shorter

than the least which could be measured on the lever system used in these experiments. It is therefore impossible to present any conductance-gap width graph for this type of gap. It can be said with certainty, however, as a result of this experiment, that if there was any conductance through the short gaps which was not due to foreign particles or to bridging, such conductance per unit area did not exceed one reciprocal ohm per square centimeter for a gap as narrow as 2×10^{-6} cm. A conduction curve for any gap narrower than this would be subject to serious criticism, as very little is known about the smoothness of any type of polished surfaces within such fine limits.

As a result of these experiments the conclusion may be drawn that appreciable electrical current can pass between surfaces of approximately 4 cm. diameter which are separated as much as 10,000 Å in air. The fact that at such a distance the current is approximately proportional to the square of the applied potential difference leads one to believe that charged dust particles in the air are playing the same part that ions do in solutions. It is thought that the dust particles become charged at one electrode, are drawn over to the other and become discharged or charged oppositely and thus return to the first electrode to repeat the cycle. Such a process seems to take place even if all ordinary precautions are taken to exclude dust from the space between the surfaces or even if the electrodes are placed in a bell jar and the air pressure reduced to 0.01 mm. of mercury for several days. In the present experiments, it seems that when the nearly flat electrodes are brought closer together some of the current is carried by invisible prominences which touch each other. In this way one can account for the fact that gradually the relation between current and voltage changes from an ionic one to an ohmic one. At very small separations the current is definitely ohmic and is probably caused by metallic bridging.

The writers are happy to take this opportunity to express their appreciation for a research grant for this work from the Bache Fund of the National Academy of Sciences.

* Now at the University of Toledo.

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THE COLOR CHANGES IN UCA PUGILATOR

BY SVEN PH. CARLSON

STOCKHOLM, SWEDEN*

Communicated July 22, 1935

During recent years it has been proved, especially through the investigations of Koller and of Perkins, that there probably is an organ in the eye-stalks of crustaceans, which by incretory means controls the color changes of the body. This organ is influenced by stimulations from the eyes. The incretory substance is similar to a hormone and brings about contraction of the chromatophores in Crangon and Palaemonetes. Kropp and Perkins (1933) have found that this hormone occurs in the eye-stalks of all crustaceans thus far investigated, some 16 species, even in those in which there are no color changes themselves. Meyer (1931) showed that this hormone is not only active in the chromatophores of crustaceans but also calls forth contraction of the color cells in fishes (Gobius and Pleuronectes). In 1933 Kropp and Perkins proved that an injection of extract from the eye-stalks of crustaceans into tadpoles of Rana clamitans brings about a strong expansion of the chromatophores in this species. Koller states (1928) that in Crangon he has succeeded in getting expansion of the chromatophores by injection of an extract from the rostral region. Perkins and Snook (1931) and Perkins and Kropp (1933), however, have not observed any effects at all from such a rostral extract.

Hanström (1931–1934) has found two different organs in the eye-stalks of crustaceans, the *X*-organ and the blood-gland, as he calls them. These are both innervated from the medulla terminalis and, to judge from their histological structure, they are secretory in function.

In the course of experiments made together with Dr. Hanström at the Marine Biological Laboratory, Woods Hole, during the summer 1935 in studying these secretions, a number of different crustaceans have been investigated for their color changes. During these researches I found that the fiddler crab, Uca pugilator, behaved in a manner quite different from all other crustaceans.

For a long time it has been known that Palaemonetes and Crangon grow dark if their eye-stalks are cut off, and this they do either in light or in darkness, on a white background or on a black one. Uca pugilator on the contrary grows pale if the eye-stalks are cut off which is in accordance with the observations of Megnsar (1912) in Uca pugnax. The dark chromatophores are maximally contracted. This can be observed especially clearly in the legs, where the chromatophore changes from a large and richly branched form to a small and punctate one. The whole animal grows pale and will remain so independently of the degree of light and

the color of the background. If an extract is made from the eye-stalks of Uca and injected into pale animals, without eyes and with maximally contracted chromatophores, they grow dark again the more rapidly the stronger the extract. An obvious effect is obtained even by an amount of extract corresponding to one-tenth of one eye-stalk. After 5 minutes one can see under the microscope how the chromatophore branches extend. Maximal expansion is obtained within 2 hours, and then the animals slowly grow pale and within 4 hours the chromatophores are maximally contracted again.

Extracts from Palaemonetes and several other crustaceans work in the same way. The effects are different only according to the strength of the extracts and all crustaceans thus far investigated have given this reaction.

Extract from the eye-stalks of Uca, when injected into Palaemonetes produces a strong contraction of the chromatophores quite in the same way as those of Palaemonetes on itself. Extracts of all other crustaceans tested acted in the same way.

In all probability we have in Uca the same common crustacean hormone, but the effect on the dark chromatophores of Uca is quite contrary to that in all other crustaceans and fishes investigated, but is exactly the same as the crustacean hormone has on frogs.

In order to find out the position of the active organ, extracts were made from different parts of the eye-stalks. The eye-stalk of Uca is especially suited to such experiments on account of its length, which makes it possible to cut it in several suitably small pieces. As test objects eyeless Ucas were first used, but later eyeless and hence quite dark Palaemonetes proved to give a more rapid and sensitive reaction.

The eye-stalks of Uca were cut transversally in three equal parts. Only the middle part proved to be active, whereas the distal and proximal parts were quite inactive.

In the same way such animals in which the distal parts of both eye-stalks had been cut off were found to keep their original dark color. If then the middle part was cut off, the animals grew pale as if the entire eye-stalks had been removed.

From these experiments it seems that the active substance is derived from the middle part of the eye-stalks of Uca. By microscopical examination I have not been able to discover the *X*-organ of Hanström in Uca pugilator. On the contrary I have found a very fully developed blood-gland, innervated from the medulla terminalis and containing eosinophilic drops. It is situated on the surface of the medulla terminalis at its distal part and thus within the active part of the eye-stalks. I have not succeeded in discovering any other innervated organs of secretion in this part of the eye-stalk.

Thus my experiments on Uca prove that the active substance is derived

from the middle part of the eye-stalk and it seems probable that the blood-gland of Hanström produces it.

I have repeated these experiments on Uca pugnax and have obtained exactly the same results.

A detailed account of this work with photographs and references will be published later, probably in Kungl. Fysiografiska Sällskapets Förhandlingar, Lund, Sweden.

* Marine Biological Laboratory, Woods Hole, Mass.

PROCESSES OF LOCALIZATION IN DEVELOPING FUNDULUS

BY JANE M. OPPENHEIMER

OSBORN ZOÖLOGICAL LABORATORY, YALE UNIVERSITY

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Localized vital staining has been used to study developing Fundulus, in order to ascertain, first, the early localization of embryonic organs in the cleaving egg, and second, the movements by which the cells forming these organs attain their final position in the embryo. The technique employed[1] varies slightly from that originated by Vogt.[2] The results show that the long axis of the embryo does not necessarily coincide with the first or second plane of cleavage, and hence that any mosaic which exists in the cleaving egg is independent of the position of the planes of cleavage. Further, the presumptive forebrain material has in some cases been traced to one or two of the central cells of the sixteen-cell stage, and part of the germ-ring to the outer half of marginal cells of the same stage. Finally, gastrulation in Fundulus, while fundamentally of the same type as in the amphibian (Vogt[3]) and the trout (Pasteels[4]), is accompanied by accessory processes not described for the other forms.

At the completion of cleavage in Fundulus, the blastoderm is a homogeneous cap of cells covering a small surface of the large uncleaved yolk. Accompanying or preceding the onset of gastrulation, the cells of the blastoderm concentrate at the periphery to form the germ-ring, leaving a central area of fewer cells. The central area is slightly eccentric, since the thickened rim of the blastoderm is wider than elsewhere at the region of the future embryonic axis. At this localized region invagination predominates, accompanied by a concentration of cells which forms the embryonic axis. This accumulation of cells is the embryonic shield, a plate of closely packed cells, which is sharply distinguished from the extra-embryonic membrane constituting the remainder of the blastoderm.

The staining experiments show that the cells whose derivatives form most of the nervous system are localized early in the embryonic shield; they

take up their position principally by an antero-posterior stretching and migration, only a small group of cells, indicated by horizontal hatching in figures 1*A*–1*C*, converging centrally. The early shield contains a central axis in which the cells are packed more closely than elsewhere; these are the cells which form most of the midbrain region (diagonal hatching in Fig. 1), and most of the hindbrain and part of the spinal cord (vertical hatching in Fig. 1). As these two groups of cells elongate during gastrulation, they receive contributions from the cells lying immediately lateral to the central axis of the early shield (horizontal hatching in Fig. 1). The material which later will form forebrain and eye (heavy stipple in Fig. 1)

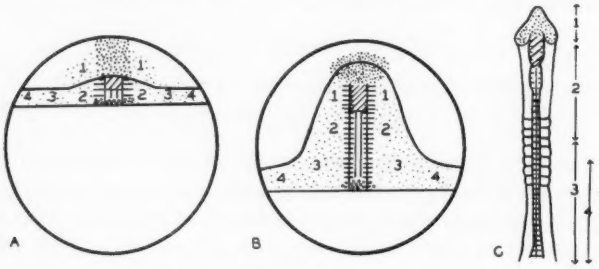

FIGURE 1

The localization of presumptive nervous tissue, mesoderm and endoderm in the early gastrula (*A*), and the middle gastrula (*B*), and the positions of these tissues in the seven-somite embryo (*C*).

The position of the nervous tissue is indicated by heavy stipple (forebrain and optic vesicles), diagonal hatching (midbrain and anterior hindbrain) and vertical hatching (posterior hindbrain and spinal cord. The cells whose position is indicated by horizontal hatching aid in the formation of midbrain, hindbrain and spinal cord.

The mesoderm is represented by the lightly stippled areas. The areas marked by the numbers 1, 2, 3 and 4 in *A* and *B* ultimately lie in the regions of the embryo indicated by the arrows accompanying diagram *C*.

The endoderm is represented by small circles.

is not part of the embryonic shield at the beginning of gastrulation, but is situated in the extra-embryonic membrane anterior to the shield and joins the shield anteriorly as gastrulation proceeds (Fig. 1*B*).

The cells constituting prospective axial mesoderm lie laterally in the early shield; and lateral to the shield in the germ-ring, and also in the extra-embryonic membrane anterior to the shield and lateral to the presumptive forebrain and eye cells (light stipple in Fig. 1). The cells constituting the lateral wings of the early embryonic shield, situated between the presumptive nervous system and the junction of the shield with the germ-ring (number 2 in Fig. 1) form the mesoderm in the posterior brain region the

anterior cord region; this group of cells, as does all the presumptive mesoderm, migrates toward the midline and also stretches antero-posteriorly. The cells in the germ-ring adjacent to the early shield (number 3 in Fig. 1) migrate axially and reach the shield when the yolk is one-half to two-thirds covered by the blastoderm. This group of cells forms a large part of the mesoderm posterior to the first few somites. Cells originally over 90° away from the embryonic axis (number 4 in Fig. 1) join the shield as the blastopore closes and aid in forming the tail-bud blastema. In mesoderm-formation, just as previously described for nervous system formation, cells originally found in the extra-embryonic membrane join the shield anteriorly (number 1 in Figs. 1*A*–1*C*).

FIGURE 2

Diagram of the tentative localization in the early gastrula of Fundulus. Dorsal view. For explanation see text.

The addition of cells to the teleostean shield at its anterior part as well as at its blastoporic lip was evidently suspected by Wilson[5] who described the growth of the Serranus embryo by "intussusception" rather than by concrescence. The occurrence of the phenomenon in Fundulus suggests that induction may occur independently of actual invagination.

The endoderm is derived from cells at the posterior end of the shield, which form the lip and migrate inward along the axis mesial to the mesoderm. The notochord has not been positively located by the vital staining method, but the contributing cells are probably situated between the areas for endoderm and nervous system. A tentative map of the early gastrula is shown in figure 2. In this diagram the vertical hatching represents the presumptive nervous system, horizontal hatching the mesoderm, heavy stipple the notochord and light stipple the endoderm. This map differs from Vogt's for the amphibian and from Pasteels' for the trout principally in the form of the presumptive nervous system area; in Fundulus this is a compact area which stretches antero-posteriorly rather than a crescentic area which converges or rotates into position.

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[2] Vogt, W., *Arch. Entwmech.*, **106**, 542–610 (1925).
[3] Vogt, W., *Ibid.*, **120**, 384–706 (1929).
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AN UNBIASED CORRELATION RATIO MEASURE

BY TRUMAN L. KELLEY

HARVARD UNIVERSITY

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The properties of the correlation ratio have been very thoroughly studied and reported upon. It has long been a necessary instrument in the study of the nature of regression. The work of Fisher[1] in 1922 made it a very precise instrument in studying the goodness of fit of second and higher degree regression lines.

It, however, lacks a certain desirable simplicity of meaning in that its value, η, obtained from a sample, differs from the population value, $\tilde{\eta}$, not only in a random manner due to the fluctuation of the particular sample, but also in a systematic manner which is a function of the number of arrays in which the data are recorded. This systematic difference between η and $\tilde{\eta}$ is, of course, well known to the expert statistician, and allowed for in his interpretations, as, for example, is automatically the case in the use of the following formula by Fisher

$$\chi^2 = (N - k)\frac{\eta^2 - R^2}{1 - \eta^2} \qquad (1)$$

where N is the number of cases in the sample, k the number of arrays in which the dependent variable is classed, η the ordinary correlation ratio, R the ratio of the standard deviation of the differences between the points upon the regression line used and the means of the arrays of the sample to the standard deviation of the dependent variable (for a linear regression line, R is simply r, the ordinary product-moment correlation coefficient), and χ^2 is the ordinary χ^2 distributed nearly in the Pearson type III manner and with a number of degrees of freedom equal to $[k - f(R)]$ in which $f(R)$ is the number of linear restrictions placed upon the frequencies in determining the regression line employed, it equaling 2 in the case in straight line regression, 3 for second degree parabolic regression, etc.

Entering a table giving probabilities for values of χ^2 with the value given by [1] and a number of degrees of freedom equal to $[k - f(R)]$ yields a value P which is the probability that if the true regression, or regression in the population, is of the form assumed, a divergence from it as great as that observed would arise as a matter of chance. Thus P, derived for χ^2, is an immediately interpretable statistic. We may note the simplicity of several of the other concepts.

N = the number of cases in the sample
k = the number of arrays

$k - f(R)$ = the number of degrees of freedom in the differences between the means of the arrays and the points on the regression line

R = an unbiased estimate of the R, which we will designate $\tilde{R}$, in the population.

However, η is not an unbiased estimate of the correlation ratio, $\tilde{\eta}$, in the population.

The intent of the present article is to provide a new correlation ratio squared, ϵ^2, which is an unbiased estimate of $\tilde{\eta}^2$; to provide its standard error; and to substitute it in place of η^2 in Fisher's formula [1]. So far as P and χ^2 are concerned, its use leads to identical results with [1].

The need to allow for the systematic effect of fine grouping for small numbers in arrays was handled by Pearson[2] in 1923, but he did not use Fisher's results of 1922. In 1906 Blakeman[3] provided a test for linearity which now, upon the basis of the later work of Pearson and Fisher, we must believe to be inadequate.

In the following treatment the notation will indicate population statistics by employing a tilda circumflex, ~; statistics which are means from a large number of samples by the caret, ^; statistics which are sample means by the macron, –; and other sample statistics by symbols having none of these circumflexes. The difference between the ^ and the ~ statistics is entirely a matter of the size of the samples entering into the average yielding the ^ statistics. With large samples every ^ statistic approaches the corresponding ~ statistic. Subscripts a, b, c, $\ldots k$ will refer to the k successive classes of the independent variable. Also subscript a will refer to any one such array. Letting v without subscript represent the variance of the dependent variable for the marginal total, and v with subscript the variance for an array, the true, or population, value of the correlation ratio is given by

$$\tilde{\eta}^2 = 1 - \frac{S\tilde{n}_a\tilde{v}_a}{\tilde{v}S\tilde{n}_a} \qquad (2)$$

wherein $\tilde{n}_a$ is the number giving the same proportion in array a as in the population, and S indicates a summation for all arrays, a, b, $\ldots$ k.

The usual or "raw" correlation ratio squared is given by the same formula dropping the circumflex. This value, η^2, is subject to a fineness of grouping error.

If the numbers in the arrays for a sample are n_a, n_b, $\ldots$ n_k giving proportions differing in a random manner from the proportions in the population, which are $\tilde{n}_a$, $\tilde{n}_b$, $\ldots$ $\tilde{n}_k$, the function,

$$\frac{Sn_a\tilde{v}_a}{Sn_a}$$

will differ by chance only from

$$\frac{S\tilde{n}_a\tilde{v}_a}{S\tilde{n}_a}.$$

Thus we may write the approximate equality

$$\tilde{\eta}^2 = 1 - \frac{Sn_a\tilde{v}_a}{\tilde{v}Sn_a}. \tag{3}$$

This is an estimate of $\tilde{\eta}^2$ having no systematic error. We now desire estimates of $\tilde{v}_a$, $\tilde{v}_b$, ... $\tilde{v}_k$, and $\tilde{v}$ having no systematic errors. As shown by "Student,"[4] these are readily available. They are, respectively,

$$\frac{n_a v_a}{n_a - 1}, \frac{n_b v_b}{n_b - 1}, \cdots \frac{n_k v_k}{n_k - 1}, \frac{Nv}{N - 1}$$

wherein $N = Sn_a$.

If we introduce these values in (3) we will obtain a formula for η^2 in which the estimated true variances for the arrays are weighted according to the number of cases in the sample in each array. Each array variance, i.e., each $\frac{n_c\ v_a}{n_a - 1}$, is an estimate of the residual variance in the dependent variable knowing the value (category) of the independent variable. However, the average of independent measures of the same thing having the least standard error is given by weighting each inversely as its variance, as early shown by Gauss.[5] The variance of

$$\frac{n_a\ v_a}{n_a - 1} \tag{4}$$

equals $\frac{n_a^2}{(n_a - 1)^2}$ times the variance of v_a, which for any distribution is as given by "Student"

$$v(v_a) = 2\tilde{v}_a^2 \frac{n_a - 1}{n_a^2} \tag{5}$$

We accordingly obtain the variance of (4)

$$\frac{2\tilde{v}_a^2}{n_a - 1}$$

Introducing the reciprocals of these as weighting factors in place of n_a, n_b, ... n_k in (3) yields

$$\epsilon^2 = 1 - \frac{S\left(\frac{n_a - 1}{2\tilde{v}_a}\right)\left(\frac{n_a\ v_a}{n_a - 1}\right)}{v\frac{N}{N - 1}\ S\frac{n_a - 1}{2\tilde{v}_a}} \tag{6}$$

It is entirely conceivable that $\tilde{v}_a \neq \tilde{v}_b \neq \tilde{v}_c$, etc., but since these are of the nature of residual variances, it would not seem violent to assume them equal, in which case (6) becomes

$$\epsilon^2 = 1 - \frac{(N-1)Sn_a v_a}{(N-k)Nv}. \tag{7}$$

Since $\eta^2 = 1 - \frac{Sn_a v_a}{Nv}$, we may write (7) as

$$\epsilon^2 = \frac{(N-1)\eta^2 + 1 - k}{N-k} = \frac{N\eta^2 - k + (1-\eta^2)}{N-k}. \tag{8}$$

Comparing results here obtained with those of Pearson in 1923[6] we find certain small but significant differences.

Pearson gives the chance η^2 value, in case true $\eta^2 = 0$, as $\frac{(k-1)}{N}$, whereas from (8) we obtain $\frac{k-1}{N-1}$, which value has also been obtained by Wishart.[7] Pearson gives η^2 corrected for fineness of grouping, in case η^2 is fairly large, as equal to $\frac{N\eta^2 - k + 3}{N-k+3}$, which differs slightly from (8).

Let us now determine the standard error of ϵ^2, given by (7).

$$v(\epsilon^2) = \left(\frac{N-1}{N-k}\right)^2 v\left(\frac{Sn_a v_a}{Nv}\right) \tag{9}$$

$$Nv = Sx^2$$

in which x is the dependent variable as a deviation from the sample mean. For array a each x may be written

$$x_a = \bar{x}_a + (x_a - \bar{x}_a)$$

where $\bar{x}_a$ is the mean for the array as a deviation from the sample mean, and, as shown by "Student," $\bar{x}_a$ is uncorrelated with $v(x_a - \bar{x}_a)$. We may therefore write

$$Sx^2 = Sn_a\bar{x}_a^2 + Sn_a v_a \tag{10}$$

Let
$$f = \frac{Sn_a v_a}{Nv}. \tag{11}$$

Then
$$v(\epsilon^2) = \left(\frac{N-1}{N-k}\right)^2 v_f \tag{12}$$

And
$$v_\epsilon = \frac{v(\epsilon^2)}{4\epsilon^2} = \frac{(N-1)^2}{4(N-k)^2\epsilon^2}\, v_f \tag{13}$$

$$f = (1 - \epsilon^2)\frac{N-k}{N-1}$$

$$= \frac{Sn_a v_a}{Sn_a v_a + Sn_a \bar{x}_a^2} \tag{14}$$

$$\frac{df}{\hat{f}} = \frac{dSn_a v_a}{Sn_a v_a} - \frac{dSn_a v_a + dSn_a \bar{x}_a^2}{Sn_a \hat{v}_a + Sn_a \bar{x}_a^2} \tag{15}$$

Employing the earlier assumption that $\bar{v}_a = \bar{v}_b = \ldots \bar{v}_k$

$$Sn_a \hat{v}_a = Sn_a \frac{n_a - 1}{n_a} \bar{v}_a = (N-k)\bar{v}_a \tag{16}$$

$$S\hat{x}^2 = (N-1)\bar{v}. \tag{17}$$

Taking logarithmic differentials of (14)

$$\frac{df}{\hat{f}} = \frac{dSn_a v_a}{(N-k)\bar{v}_a} - \frac{dNv}{(N-1)\bar{v}}. \tag{18}$$

Squaring, summing and dividing by the number of samples

$$\frac{vf}{\hat{f}^2} = \frac{S2n_a^2 \bar{v}_a^2 \dfrac{n_a - 1}{n_a^2}}{(N-k)^2 \bar{v}_a^2} + \frac{2N^2\bar{v}^2 \dfrac{N-1}{N^2}}{(N-1)^{2-2}} - \frac{2S2n_a^2 \bar{v}_a^2 \dfrac{n_a - 1}{n_a^2}}{(N-k)(N-1)\bar{v}_a \bar{v}}$$

$$= \frac{2}{N-k} + \frac{2}{N-1} - \frac{4(1-\eta^2)}{N-1}. \tag{19}$$

From (12)

$$v(\epsilon^2) = \frac{(1-\epsilon^2)^2(N-k)^2(N-1)^2}{(N-1)^2\,(N-k)^2}\left\{\frac{2}{N-k} - \frac{2-4\epsilon^2}{N-1}\right\}$$

$$v(\epsilon^2) = \frac{(1-\epsilon^2)^2}{N-1}\left\{\frac{2(k-1)}{N-k} + 4\epsilon^2\right\} \tag{20}$$

$$\sigma_{\epsilon^2} = \frac{1-\epsilon^2}{\sqrt{N-1}}\left\{\frac{2(k-1)}{N-k} + 4\epsilon^2\right\}^{1/2} \tag{21}$$

$$v_\epsilon = \frac{v(\epsilon^2)}{4\epsilon^2} = \frac{(1-\epsilon^2)^2}{4(N-1)\epsilon^2}\left\{\frac{2(k-1)}{N-k} + 4\epsilon^2\right\} \cdot \cdot \begin{Bmatrix}\epsilon^2 \text{ not} \\ \text{small}\end{Bmatrix} \tag{22}$$

$$\sigma_\epsilon = \frac{1-\epsilon^2}{2\epsilon\sqrt{N-1}}\left\{\frac{2(k-1)}{N-k} + 4\epsilon^2\right\}^{1/2} \cdot \cdot \begin{Bmatrix}\epsilon^2 \text{ not} \\ \text{small}\end{Bmatrix} \tag{23}$$

Formulas (22) and (23) are not usable formulas in case ϵ is small, as certain higher order terms have been neglected. Formulas (20) and (21) are

generally useful except when $\frac{1}{N}$ is not small in comparison with $\frac{1}{\sqrt{N}}$.

To obtain a test for goodness of fit we will make a substitution in (1). In place of η^2 we will substitute ϵ^2 as given by (8). This yields

$$\chi^2 = \frac{(N-k)(\epsilon^2 - R^2) + (k-1)(1-R^2)}{1-\epsilon^2} \qquad (24)$$

$\{k - f(R)$ degrees of freedom$\}$

A second substitution is called for in that R^2 will differ in a slightly systematic manner from $\tilde{R}^2$. We have

$$R = \hat{R} + \delta R$$

and

$$\hat{R}^2 = \tilde{R}^2 + v(R).$$

If we define ρ^2 by the equation

$$R^2 = \rho^2 + v(R)$$

the quantity $(\epsilon^2 - \rho^2)$ and not $(\epsilon^2 - R^2)$ is a quantity whose mean value would equal zero in the case of correctness of the assumption as to the nature of the regression. However, the difference between $(\epsilon^2 - \rho^2)$ and $(\epsilon^2 - R^2)$ will ordinarily be so small that unappreciable gain in interpretability of the elements entering into (24) would result from incorporating ρ^2 instead of R^2 into the formula. As (24) stands, the magnitude of $(\epsilon^2 - R^2)$ is itself a good indication of goodness of fit. In particular, if $(\epsilon^2 - R^2) < 0$, an excellent fit is indicated without calculating χ^2 or P. If $(\epsilon^2 - R^2)$ is much less than zero, one should be skeptical of the arithmetical accuracy of his computations or of the logical soundness of some step in his treatment.

[1] R. A. Fisher, "The Goodness of Fit of Regression Formulae and the Distribution of Regression Coefficients," *Jour. Roy. Stat. Soc.*, **85**, Part IV, July (1922).

[2] "On the Correction Necessary for the Correlation Ratio," *Biometrika*, **14** (1923).

[3] John Blakeman, "On Tests for Linearity of Regression in Frequency Distributions," *Ibid.*, **4** (1905–06).

[4] "The Probable Error of a Mean," *Ibid.*, **6**, No. 1 (1908).

[5] See Kelley, Truman L., *Statistical Method*, 1923, Formula [309].

[6] "On the Correction Necessary for the Correlation Ratio," *Biometrika*, **14** (1923).

[7] John Wishart, "A Note on the Distribution of the Correlation Ratio," *Ibid.*, **24** (1932).